We might conclude from this example that little is gained from going to a GP solution framework, but the advantages of GP become obvious in more complex problems.

Table 13.1. Solution to Single Goal Model

		$c_j \rightarrow$	0	0	−1	0	0	0
Basic var.	C_B	X_B (solution)	X_1	X_2	d^-	d^+	S_1	S_2
s_1	0	20	10	0	0	0	1	−1
x_2	0	4	1	1	0	0	0	1/10
d^-	−1	680	−40	0	1	−1	0	−8
		$z = -680$	40	0	0	1	0	8 $\leftarrow \Delta_j$

Now consider an example where there exists a single goal that cannot be solved by linear programming.

Example 2. *A company is considering the allocation of Rs. 150,000 advertising budget to two magazines (A and B). Rated exposures per hundred rupees of advertising expenditures are 1,000 and 750, respectively for the two magazines; and it has been forecast that on the average Rs. 10 in sales results from each advertisement exposure. Management has decided that no more than 75% of the advertising budget can be expended in magazine A. The company has indicated that it would like to achieve exactly 1·5 million exposures from its advertising program. Management's objective is to allocate its money to advertising in such a way that sales (Rs.) are maximized.*

Solution. The LP formulation for the problem is :

$$\text{Maximize : } \quad z = 10{,}000\,x_1 + 7{,}500\,x_2 \qquad \text{...(13.8)}$$
$$\text{Subject to : } \quad x_1 + x_2 \le 1{,}500 \qquad \text{...(13.9)}$$
$$x_1 \le 1{,}125 \qquad \text{...(13.10)}$$
$$1{,}000\,x_1 + 750\,x_2 = 1{,}500{,}000 \qquad \text{...(13.11)}$$
$$x_1, x_2 \ge 0,$$

where x_1 = hundreds of rupees spent on advertising in magazine A

x_2 = hundreds of repees spent on advertising in magazine B.

If we attempt to solve this problem by simplex method, the results would be an infeasible solution. With an LP structure, no feasible solution results because it is impossible to achieve 1,500,000 exposures without violating the constraint (13.10).

If we examine the problem closely, we can see that in maximizing sales the company is in essence maximizing exposures, since each magazine exposure results in Rs. 10 in sales. Since company has set a target goal of 1,500,000 exposures, achieving this goal also means company is maximizing sales. Therefore, expressed in a GP framework, the problem is :

$$\text{Minimize : } \quad z = d^- + d^+ \qquad \text{...(13.12)}$$
$$\text{Subject to : } \quad x_1 + x_2 \le 1{,}500 \qquad \text{...(13.13)}$$
$$x_1 \le 1{,}125 \qquad \text{...(13.14)}$$
$$1{,}000\,x_1 + 750\,x_2 + d^- - d^+ = 1{,}500{,}000 \qquad \text{...(13.15)}$$
$$x_1, x_2, d^-, d^+ \ge 0.$$

As the problem was originally stated, company is interested in achieving the exposure goal exactly. Both deviational variables, therefore, are included in the objective function.

Using the simplex method to solve the GP formulation of companies single-goal advertising problem results in the following : $x_1 = 1{,}125$, $x_2 = 375$, $d^+ = 0$, and $d^- = 93{,}750$. This indicates that company under-achieved its exposure goal of 1.5 million by 93,750. The GP model does not result in achieving the impossible (management is still unable to achieve 1.5 million exposures) but the model does provide a *feasible mathematical solution.* In the next section, we will find that by including deviational variables in the budget constraint, (13.13) and establishing a second goal of attempting to stay within the budget, the additional budget rupees required to meet the exposure goal can be identified.

13.3-2 Multiple-Goal Models

In this section we will examine multiple-goal models. Three types of models exist : *multi-goal with equal (no) priorities, multi-goal with priorities,* and *multi-goal with priorities and weights.* In the real world, the third formulation is most useful; however, we can gain a better understanding of the concept of priorities and weights by examining each type of model.

Multiple Goals with Equal (No) Priorities

The multigoal equal-priority model is easy to deal with mathematically but is the least practical of the three formualtions. Some actual cases may exist where all goals have equal priorities, but the output from an equal-priority model should be examined carefully for compatibility. Consider the following problem.

Example 3. *Modify the single-goal production problem in the previous section so that in addition to the profit goal, at least two units of each type of product are desired. Management considers this second goal to be equally as important as the profit goal, which is maximizing profit. Under these conditions, management is saying that a deviation of Re. 1 from the profit goal is considered to be equal to a one-unit deviation from the production goal.*

Solution : The GP formualtion for the problem is :

$$\begin{array}{llr} \text{Minimize:} & z = d_1^- + d_2^- + d_3^- & \text{...(13.16)} \\ \text{Subject to:} & 20x_1 + 10x_2 \le 60 & \text{...(13.17)} \\ & 10x_1 + 10x_2 \le 40 & \text{...(13.18)} \\ & 40x_1 + 80x_2 + d_1^- - d_1^+ = 1000 & \text{...(13.19)} \\ & x_1 + d_2^- - d_2^+ = 2 & \text{...(13.20)} \\ & x_2 + d_3^- - d_3^+ = 2 & \text{...(13.21)} \\ & x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \ge 0 & \end{array}$$

Since the deviational variables in the objective function have no priorities, we can continue to use the standard simplex algorithm to solve the problem. Table 13.2 is the final tableau that results from applying the simpelx method. The solution is $x_1 = 0, x_2 = 4, s_1 = 20, s_2 = 0, d_1^- = 680$, $d_1^+ = 0, d_2^- = 2, d_2^+ = 0$, $d_3^- = 0, d_3^+ = 2$, and $z = 682$. In terms of production ($x_1 = 0, x_2 = 4$) and profits (Rs. 1000 – Rs. 680), this solution is identical to the single-goal model. The 682 value for z indicates that we missed the profit goal by 680 and one of the production goals by 2. (The z value of 682 reflects the sum of the extent by which we missed *all* goals.) The model did not achieve the production goals of at least two units of x_1 and two units of x_2 because no priorities were specified for the goals; the model simply sought to minimize the *sum* of the deviations for *all* goals.

Table 13.2. Final Table of Example with multiple Goals and No Priorities

		$c_j \rightarrow$	0	0	–1	0	–1	0	–1	0	0	0	
Basic var.	C_B	X_B	X_1	X_2	d_1^-	d_1^+	d_2^-	d_2^+	d_3^-	d_3^+	S_1	S_2	
s_1	0	20	10	0	0	0	0	0	0	0	1	–1	
d_3^+	0	2	1	0	0	0	0	0	–1	1	0	1	
d_1^-	–1	680	–40	0	1	–1	0	0	0	0	0	–8	
d_2^-	–1	2	1	0	0	0	1	–1	0	0	0	0	
x_2	0	4	1	1	0	0	0	0	0	0	0	1	
	$z = -682$		39	0	0	1	0	1	1	0	0	8	$\leftarrow \Delta_j$

13.3-3 Multiple-Goals with Priorities

When management has multiple-goals, they will most likely have some priority scale for the goals. Goal programming provides for the preferential ordering of goals through the use of priority coefficients (P 's). All goals (deviational variables) that have a top or first priority are assigned an objective function value of P_1, the goals considered to be second in priority are assigned a P_2 value, and this process is continued until all goals have been ranked. The coefficient, P_1, P_2, etc., are not parameters or variables. They do not, in general, assume a numerical value, they simply represent levels for the priorities. Since the priority coefficients appear in the objective function, the standard simplex algorithm cannot be used to solve such problems. However, the simplex method can be modified to handle the priority coefficients and assure that the deviations for the first-priority goals are minimized before lower priorities are examined, and so on.

The use of priority goals can be demonstrated by considering the production problem just examined.

Example 4. *Management has established the following goal priorities :*

P_1 *(priority 1) : To meet production goals of 2 units for each product.*

P_2 *(priotity 2) : To maximize profits.*

Solution : The model for this problem would be identical to the previous no-priority model [equations (13.16) – (13.21)] with the exception of the addition of priority coefficients in the objective function.

The revised model is

Minimize : $$z = P_1 d_2^- + P_1 d_3^- + P_2 d_1^- \qquad (13.22)$$

Subject to :

$$20x_1 + 10x_2 \leq 60 \qquad (13.23)$$

$$10x_1 + 10x_2 \leq 40 \qquad (13.24)$$

$$40x_1 + 80x_2 + d_1^- - d_1^+ = 1000 \qquad (13.25)$$

$$x_1 + d_2^- - d_2^+ = 2 \qquad (13.26)$$

$$x_2 + d_3^- - d_3^+ = 2 \qquad (13.27)$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

The standard simplex algorithm cannot be used to solve this problem since priority indexes are in the objective function. We can, however, use a *modified GP simplex algorithm.* (see **sec. 13.5-3,** page 417)

The solution to the problem of multiple priority goals is : $x_1 = 2$, $x_2 = 2$, $s_1 = 0$, $s_2 = 0$, $d_1^- = 760$, $d_1^+ = 0$, $d_2^- = 0$, $d_2^+ = 0$, $d_3^+ = 0$, $d_3^- = 0$ and $z = 760\ P_2$. With respect to achievement of the goals the following occurred :

Goal 1, (P_1), is achieved : Two units of x_1 and 2 units of x_2 are produced.

Goal 2, (P_2), is not achieved : The total contribution to profits is Rs. 240 (that is, Rs. 1000–760); the goal is under-achieved by Rs. 760 (that is, $d_1^- = 760$).

Comparing this result with the results of the previous goal (no-priority) model, it can be seen that management must sacrifice profits (Rs. 240 versus 320) in order to meet the goal of producing at least 2 units of each product.

A second example demonstrating the use of priority ranking of goals can be given by modifying the problem [model (13.12) – (13.15)]

Example 5. *A company had a single goal of achieving 1.5 million exposures but because of budget constraints was unable to achieve the goal. If we modify the problem so that the budget constraint is set as a goal, we can determine the budget requirements necessary to meet the exposure goal. Specifically the goals are :*

P_1 *(priority 1) : To achieve an exposure goal of exactly 1.5 million exposures.*

P_2 *(priority 2) : To minimize the budget requirements.*

Solution. The model for the problem would require replacing constraint (13.13) with the following goal :

$$x_1 + x_2 + d_2^- - d_2^+ = 1{,}500 \qquad (13.28)$$

where d_2^- is the amount by which the Rs. 150,000 is under-spent and d_2^+ is the amount of over-expenditure. The complete model is :

Minimize : $$z = P_1 d_1^- + P_1 d_1^+ + P_2 d_2^+ \qquad (13.29)$$

Subject to :

$$x_1 + x_2 + d_2^- - d_2^+ = 1{,}500 \qquad (13.30)$$

$$x_1 \leq 1{,}125 \qquad (13.31)$$

$$1{,}000\, x_1 + 750 x_2 + d_1^- - d_1^+ = 1{,}500{,}000 \qquad (13.32)$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+ \geq 0$$

The solution to the problem is : $x_1 = 1125, x_2 = 500$, $d_1^- = 0$, $d_1^+ = 0$, $s_1 = 0$, $d_2^- = 0$, $d_2^+ = 125$, and $z = 125\, P_2$, where s_1 is a slack variable associated with constraint (13.31). With respect to achievement of the goals, the following results were achieved :

Goal 1, *(P_1) is achieved :* 1.5 million exposures have resulted; 1,125,000 exposures result from spending Rs. 112,500 for advertisements in magazine A, and 375,000 exposures result from spending Rs. 50,000 for advertisements in magazine B.

Goal 2, *(P_2), is exceeded :* The second goal is exceeded by 125; in terms of actual expenditure this means that the advertising budget is exceeded by Rs. 12,500. This is the additional expenditure necessary to achieve 375,000 exposures in order to meet the 1.5 million exposures that result in goal 1.

13.3-4. Multiple Goals with Priorities and Weights

Both problems in the preceding section can be said to have multiple goals with the same priority index [refer to P_1 goals in equations (13.22) and (13.29)]. Sometimes we may wish to give some of the equally ranked goals more importance than others. If this is the case, a ***differential weight*** is used to reflect the difference of importance within the same priority level. For example, suppose that profits and overtime for a given problem both having the same priority rank. If no weights are assigned, the decision maker is implicitly stating that a Re 1 deviation from the profit goal is equal in importance to one hour of overtime. If this is not true, then weights can be assigned that reflect the proper relationship. If the decision maker decides that 6 hours of overtime is equivalent to Re.. 1 of profits, then a 6 to 1 ratio of weights would be employed.

To demonstrate, let us again consider the production example in model (13.22) – (13.27). Assume we modify the goals of the problem slightly.

Example 6. *Instead of having a goal of producing 2 units of each type of product, we will set a goal of producing a minimum of 4 units of product 1 and 6 units of product 2. Since product 2 contributes twice as much to profits as proudct 1, we should produce product 2 before producing product 1. Overtime will be required in producing either quantity of the specified products. [Note that in model (13.22) – (13.27) we used all the production capacity in producing 2 units of each product.] We will assume that 50 hours of overtime is available. Suppose we set the following priorities for goal attainment :*

P_1 (priority 1) : Limit total overtime in the two production operations to 50 hours.

P_2 (priority 2) : Meet minimum production goals of 4 units of product 1 and 6 units of product 2. Use differential weights of 1 and 2, respectively, since these reflect the "weighted contribution to profits" of Rs. 40 and Rs. 80.

P_3 (priority 3) : Maximize profits.

Solution. Since overtime is required in this modified problem, it will be necessary to add deviational variables to constraints (13.23) and (13.24). We must also add a goal constraint to reflect the goal of limiting overtime to 50 hours. To allow comparisons with the previous model we will use the previous deviational variable numbers. The modified model would appear as follows :

$$\text{Minimize: } \quad z = P_1 d_6^+ + 1P_2 d_2^- + 2P_2 d_3^- + P_3 d_1^- \qquad (13.33)$$

$$\text{Subject to: } \quad 20x_1 + 10x_2 + d_4^- - d_4^+ = 60 \qquad (13.34)$$

$$10x_1 + 10x_2 + d_5^- - d_5^+ = 40 \qquad (13.35)$$

$$40x_1 + 80x_2 + d_1^- - d_1^+ = 1000 \qquad (13.36)$$

$$x_1 + d_2^- - d_2^+ = 4 \qquad (13.37)$$

$$x_2 + d_3^- - d_3^+ = 6 \qquad (13.38)$$

$$d_4^+ + d_5^+ + d_6^- - d_6^+ = 50 \qquad (13.39)$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_5^+, d_6^-, d_6^+ \geq 0$$

The d_4^+ and d_5^+ represent, respectively, the overtime required in operations 1 and 2. The d_6^- deviational variable reflects the extent to which the 50 hours of total overtime is under-utilized. A d_6^+ variable is included in goal constraint (13.39) to reflect the possibility of exceeding the 50 hours.

The solution to the modified problem is : $x_1 = 1, x_2 = 6$, $d_4^- = 0$, $d_4^+ = 20$, $d_5^- = 0$, $d_5^+ = 30$, $d_1^- = 480$, $d_1^+ = 0$, $d_2^- = 3$, $d_2^+ = 0$, $d_3^- = 0$, $d_3^+ = 0$, $d_6^+ = 0$, $d_6^- = 0$, and $z = 3P_2 + 480P_3$.

Note. *Goal 1, (P_1), is achieved* : overtime is exactly 50 hours; 20 overtime hours, (d_4^+) were used in operation 1; and 30 (d_5^+) hours were used in operation 2.

A portion of goal 2, (P_2), was met : Six units of product 2 were produced since this portion of the goal had the largest differential weight, but only one unit of product 1 was produced before the overtime limit on goal 1 occurred.

Goal 3, (P_3), was not achieved : The total contribution to profits is Rs 520 (that is, Rs. 1000 – Rs. 480). The goal is under-achieved by Rs. 480 (that is, d_1^- = Rs. 480).

The results from this revised model indicate that additional profits can be achieved by going to overtime, but the production goals cannot be met since overtime is limited to 50 hours. Because of the differential weights used on the production goals, all 6 units of product 2 were produced before producing any units of product 1. Had the weights been reversed, the 50 hours of overtime plus regular time would have been used to produce 4 units of product 1 and 1.5 units of product 2.

The importance of the ordering (ranking) of priorities can easily be demonstrated if we reverse the first two goals for the revised model. Assume that goal 1 (P_1) is to produce 4 units of product 1 and 6 units of product 2 and goal 2 (P_2) is to limit total overtime to 50 hours. We can accomplish this by modifying equation (13.33). The new objective function would be

$$\textit{Minimize:} \qquad z = 1P_1d_2^- + 2P_1d_3^- + P_2d_6^+ + P_3d_1^- \tag{13.40}$$

The solution to this problem is : $x_1 = 4$, $x_2 = 6$, $d_4^- = 0$, $d_4^+ = 80$, $d_5^- = 0$, $d_5^+ = 60$, $d_1^- = 360$, $d_1^+ = 0$, $d_2^- = 0$ $d_2^+ = 0$, $d_3^- = 0$, $d_3^+ = 0$, $d_6^- = 0$, $d_6^+ = 90$, and $z = 90P_2 + 360\,P_3$. Comparing these results with those from the previous model, (13.33) – (13.39), it can be seen that because the production goals have top priority, both production levels are met ($x_1 = 4$, $d_2^- = 0$, and $x_2 = 6$, $d_3^- = 0$). But this occurs at a cost of an additional 90 hours of overtime ($d_6^+ = 90$). This means that 140 overtime hours are required in order to meet the production goals.

Yet another solution would result if the profit goal were set at the priority 1 level while the production and overtime goals had lower priorities. Oftentimes, exploring the output for different goal priorities will give a *sensitivity of goal attainment.*

Q. 1. What is goal programming. State clearly its assumptions.

2. Identify the major differences between linear programming and goal programming.

3. Explain the terms : (i) Deviational variables (ii) Preemptive priority factors (iii) Differential weights.

4. State some problem areas in management where goal programming might be applicable.

5. Explain the difference between cardinal value and ordinal value.

6. Under what cercumstances cardinal weights can be used in the objective function of a goal programming model ? What happens if cardinal weights are attached to all priorities in the objective function of goal programming model ?

7. What is goal programming ? Why are all goal programming problems minimization problems ?
Why altering the goal priorities result in a different solution to a problem ? Explain.

8. "Goal programming appears to be the most appropriate flexible and powerful technique for complex dicision problems involving multiple conflicting objectives" Discuss.

9. Define goal programming. **[IGNOU (MCA II) 2000]**

13.4. FORMULATION OF GOAL PROGRAMMING MODELS

The previous section highlights the concept of priority coefficients (the P's) and differential weights (the w's) and identifies the role each plays in a goal programming model. In this section, we will concentrate on formulating GP models. Both *single period* and *multiperiod* problems are described.

Before examining these examples, three summary comments should be made regarding the formulation of GP models. First, two types of variables will be a part of any formulation : (1) *decision variables* (the x's), and (2) deviational variables (the d^+s and d^-s). Second, two classes of constraints can exist in a given GP model : (1) *structural constraints,* which are generally considered environmental constraints not directly related to goals; and (2) *goal constraints,* which are directly related to goals. [In model (13.29) – (13.32), constraints (13.30) and (13.32) are goal constraints while (13.31) is a structural constraint.] Finally, while in most cases a goal constraint will contain both an under-achievement (d^-) and an over-achievement (d^+) deviational variable, even when both do not appear in the objective function, it is not mandatory that both be included. Omission of either type of deviational variable in the goal constraint however, **bounds** the goal in the direction of the omission. That is, omission of d^+ places an upper bound on the goal, while omission of d^- forces a lower bound on the goal. Thus, omission of deviational variables on goals of lower priority can limit achievement of high-priority goals.

Assuming that there are m goals, p structural constraints, n decision variables, and K priority levels, the general model can be expressed as follows :

13.4–1. The General Model

$$\textit{Minimize:} \quad z = \sum_{i=1}^{m} \sum_{k=1}^{K} P_k \left(w_{i,k}^{+} d_i^{+} + w_{i,k}^{-} d_i^{-}\right) \tag{13.41}$$

$$\textit{Subject to:} \quad \sum_{j=1}^{n} a_{ij} x_j + d_i^- - d_i^+ = b_i \quad \text{for } i = 1, \ldots, m \tag{13.42}$$

$$\sum_{j=1}^{n} a_{ij} x_j \ (\leq\ =\ \geq) \quad b_i \quad \text{for } i = m+1, \ldots, m+p \tag{13.43}$$

$$x_j, d_i^+, d_i^- \geq 0 \quad \text{for } j = 1, \ldots, n;\ i = 1, \ldots, m \tag{13.44}$$

where P_k = the priority coefficient for the kth priority

$w_{i,k}^{+}$ = the relative weight of the d_i^+ variable in the kth priority level

$w_{i,k}^{-}$ = the relative weight of the d_i^- variable in the kth priority level

13.5. METHODOLOGY OF SOLUTION PROCEDURE

Up to this point in the chapter we have developed the general structure of goal programming models and have identified the process for formulating these models. In this section, we will discuss the methodology for solving GP problems. Two solution procedures are identified : (1) a graphical method, and (2) in GP algorithm. The purpose in presenting the graphical method is to give the reader a better understanding of how goal programming procedure works. The purpose in presenting the GP algorithm is to provide a table procedure that will handle problems with more than two decision variables. The table can be used to determine when specific goals are achieved and what trade-offs occurred in achieving the goals. It will be shown that the GP algorithm is a modification of the standard simplex method.

The output from computerized version of the GP algorithm can be seen to understand the procedure.

13.5-1 Graphical Solution of GP Problems

A graphical approach can also be used to solve GP problems with two decision variables like LP problems. Following procedural steps are employed in the process, after the problem has been formulated.

Step 1. Graph or plot all *structural constraints* and identify the feasible region. If no structural constraints exist, the feasible region is that area where both x_1 and x_2 are ≥ 0 (the nonnegative quadrant).

Step 2. Graph the lines corresponding to the *goal constraints.* This is accomplished by setting the deviational variables in the goal constraint to zero and plotting the resulting equation.

Step 3. Identify the top-priority solution. This is accomplished by determining the point or points within the feasible region that satisfy the highest priority goal.

Step 4. Move to the goal having the next highest priority and determine the "best" solution(s), such that this "best" solution does not degrade the solution(s) already achieved for goals of higher priority.

Step 5. Repeat step 4 until all priority levels have been investigated.

To discuss the graphical procedure, consider the following problem :

Example 7. *A Camera Company manufactures two types of 35 mm cameras. The production process for manufacturing the cameras is such that two departmental operations are required. To produce their standard camera requires 2 hours of production time in department 1 and 3 hours in department 2. To produce their deluxe model requires 4 hours of production time in department 1 and 3 hours in department 2. Currently, 80 hours of labour are available each week in each of the departments.. This labour time is a somewhat restrictive* *factor since the company has a general policy of avoiding overtime, if possible. The manufacturer's profit on each standard camera is Rs. 30, while the profit on the deluxe model is Rs. 40. Management has set the following goals :*

P_1 *(priority 1) : Avoid overtime operations in each department.*

P_2 *(priority 2) : Prior sales-records indicate that, on the average, a minimum of 10 standard and 10 deluxe cameras can be sold weekly. Management would like to meet these sales goals. Since production time may limit producing this number of each camera, and since the deluxe camera has a higher profit margin, the sales goals should be weighted by the profit contribution for the respective cameras, i.e. Rs. 30 for the standard camera, Rs. 40 for the deluxe camera. (We could also use, weights of 3 and 4 since they have the same ratios as the profit contributions.)*

P_3 *(priority 3) : Maximize profits.*

Solution. The GP model for the problem is

$$\text{Minimize}: \quad z = (P_1 d_1^+ + P_1 d_2^+) + (30P_2 d_3^- + 40P_2 d_4^-) + P_3 d_5^- \tag{13.45}$$

$$\text{Subject to}: \quad 2x_1 + 4x_2 + d_1^- - d_1^+ = 80 \tag{13.46}$$

$$3x_1 + 3x_2 + d_2^- - d_2^+ = 80 \tag{13.47}$$

$$x_1 + d_3^- - d_3^+ = 10 \tag{13.48}$$

$$x_2 + d_4^- - d_4^+ = 10 \tag{13.49}$$

$$30x_1 + 40x_2 + d_5^- - d_5^+ = 1200 \tag{13.50}$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_5^+ \geq 0$$

The value 1200 for the right-hand side of equation (13.50) is an arbitrarily high profit goal.

Given that the model has been formulated, we can proceed with graphical method.

Step 1 : Graph all structural constraints. Since no structural constraints exist, the feasible region is the nonnegative quadrant, *i.e.*, the feasible region is the area where $x_1 \geq 0$ and $x_2 \geq 0$.

Step 2 : Graph all goal constraints. By setting the deviational variables to zero in each constraint, we can plot the resulting goal equations. These are identified in Fig 13.1 as GC_1 GC_2, ..., GC_5.

The effect of a deviational variable on a goal equation is to shift the equation so that the new equation is parallel to the original equation. The value of the deviational variable reflects the degree of shift. The directed arrows in Fig 13.1 emanating from each goal equation reflect the impact of the deviational variables on the equation. The particular deviational variables to be minimized in the objective function have been circled.

Step 3 : Identify the top-priority solution. The d_1^+ and d_2^+ deviational variables have a priority coefficient of P_1 in the objective function. Therefore, goal constraints 1 and 2 must be considered first. The objective is to minimize d_1^+ and d_2^+. If we set d_1^+ to zero, the area of feasibility (in Fig 13.1) is that area *below* GC_1 equation. If we set d_2^+ to zero, the area of feasibility is the area below GC_2 equation. The feasible area common to both is that below the line AC and CE. Any point in this area will satisfy the condition $d_1^+ = 0$ and $d_2^+ = 0$.

Note that it is incorrect to conclude that point C is the *optimal* solution to the top-priority, since the entire feasible area satisfies the condition $d_1^+ = 0$ and $d_2^+ = 0$.

Step 4 : Move to next highest priority and determine "best" solution, without violating prior goals. The d_3^- and d_4^- deviational variables have a priority coefficient of P_2 in the objective function. Therefore, we examine these goal constraints next. Since $P_2 d_4^-$ factor has a differential weight of 40 while the $P_2 d_3^-$ factor has a weight of 30, we should seek to drive d_4^- to zero before considering d_3^-. In this example, we can drive both d_4^- and d_3^- to zero and still remain within the the feasible region identified in *Step 3*. However, driving these variables to zero, we will reduce the feasible region. The new feasible region is identified by the line BC, CD, DG and GB. Any point in this feasible area will satisfy the condition $d_1^+ = 0$, $d_2^+ = 0$, $d_3^- = 0$, $d_4^- = 0$.

Step 5 : Repeat Step 4 until all priorities are investigated. The last priority level is P_3 ; the deviational variable associated with this priority is d_5^-. We have been able to drive the first four deviational variables to zero, which means that we have met the goal with which these variables are associated. If we can drive d_5^- to zero, we will have achieved all the goals. However, if we examine Fig 13.1, we note that the goal equation GC_5 lies outside the current feasible region. Since GC_5 is not feasible, the P_3 level goal can only be achieved at the expense of goals with higher priorities.

In order to have an acceptable solution that does not destroy the achievement of goals with a higher priority, d_5^- must be *positive.* Obviously, we would like d_5^- to be as small as possible. If we draw lines parallel to GC_5 until we contact the feasible region (*BCDGB*), we can identify the solution that best satisfies our objective of minimizing d_5^-. This is a point *C* on the feasible region. Since goal equations 1 and 2 (*i.e.,* GC_1 and GC_2) intersect at point *C*, we can solve these equations to find the final solution for decision variables : x_1 = 13.33 and x_2 = 13.33.

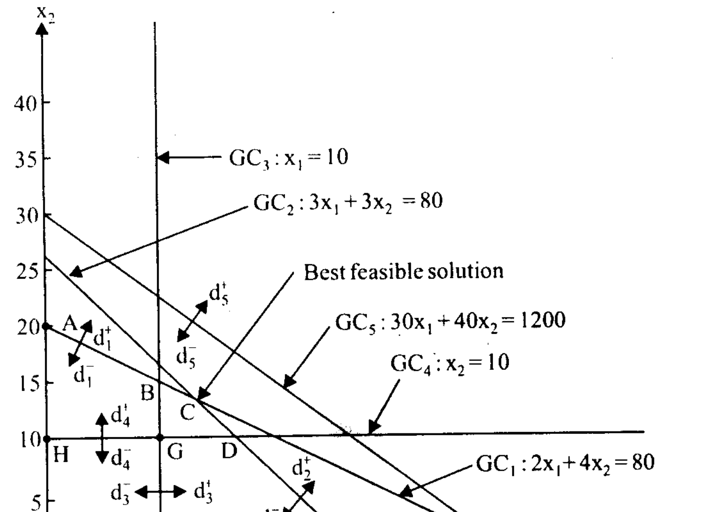

Fig. 13.1. Graphical Representation.

To determine the solution values for the deviational vairables we must employ both, the solution graph and the original goal constraints, equations (13.45) – (13.50). In Fig. 13.1, the values for deviational variables are : $d_1^- = 0$, $d_1^+ = 0$, $d_2^- = 0$, $d_2^+ = 0$, $d_3^- = 0$, $d_4^- = 0$, $d_5^+ = 0$

Substituting this information, along with the values for x_1 and x_2 , into the respective goal constraints, we can determine that $d_3^+ = 3.33$, $d_5^- = 266.67$, $d_4^+ = 3.33$.

Using the graphical procedure, the solution to the problem is as follows :

By producing 13.33 units/week of each camera, the company can achieve its first two objectives and can maximize profits. Exactly 80 hours will be used in each departmental operation, and profits will be Rs. 933.33. The company will underachieve its objective of Rs 1200 profit by Rs. 266.67.

Note : Note that this solution calls for the production of a fractional number of units. In implementing this solution, fractional units would be carried forward into the next time period (week). To find an integer solution for the problem, we would have to restrict the solution process in much the same manner as in solving an LP problem for integer variables. Some research has been devoted to developing integer-goal programming algorithms.

Example 8. *A manufacturing firm produces two types of products A and B. According to past experience, production of either product A or B requires an average of one hour in the plant. The plant has a normal production capacity of 400 hours a month. The marketing department of the firm reports that because of* *limited market, the maximum number of products A and B that can be sold in a month are 240 and 300 respectively. The net profit from the sale of products A and B are Rs. 800 and Rs. 400, respectively. The manager of the firm has set the following goals arranged in the order of importance (preemptive priority factors.)*

P_1 : *He wants to avoid any under-utilization of normal production capacity.*

P_2 : *He wants to sell maximum possible units of products A and B. Since the net profit from the sale of product A is twice the amount from product B. Therefore, the manager has twice as much desire to achieve sales for product A as for product B.*

P_3 : *He wants to minimize the overtime operation of the plant as much as possible.*

Formulate and solve the given problem by graphical method of goal programming.

Solution : Let x_1 and x_2 be the number of units of product A and B to be produced respectively. Since overtime operation is allowed, the plant capacity constraint can be expressed as

$$x_1 + x_2 + d_1^- - d_1^+ = 400 \qquad \textit{(capacity constraint)}$$

where d_1^- is the under-utilization (idle time) of production capacity, and d_1^+ is the overtime operation of the normal production capacity.

Since the sales goals are the maximum possible sales volume, positive deviations will not appear in the sales volume. Thus, the sales constraints can be expressed as

$$x_1 + d_2^- = 240 \text{ and } x_2 + d_3^- = 300 \qquad \textit{(sales constraints)}$$

where d_2^- and d_3^- are the under-achievements of the sales goal for product A and B, respectively.

Thus, the goal programming model becomes :

$$\textit{Minimize}: \; z = P_1 d_1^- + P_2 (2d_2^- + d_3^-) + P_3 d_1^+,$$

$$\textit{Subject to}: \; x_1 + x_2 + d_1^- - d_1^+ = 400$$

$$x_1 + d_2^- = 240$$

$$x_2 + d_3^- = 300$$

and $\quad x_1, x_2, d_1^-, d_2^-, d_3^-, d_1^+ \geq 0.$

Step 1. The first step in solving the problem by graphical method is to plot all the goal constraints on a graph as shown in the figure. Since the under-utilization and over-utilization of the plant capacity are

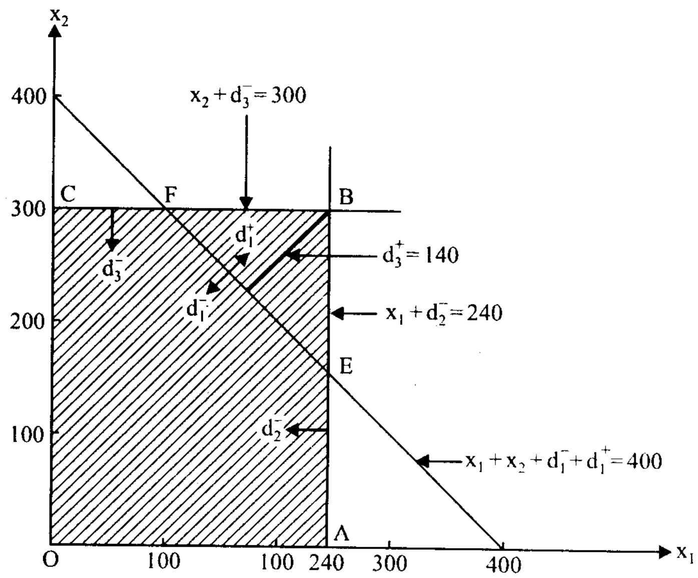

Fig. 13.2

permissible, both the deviational variables d_1^- and d_1^+ are indicated by the arrows and are shown in the figure. Similarly, d_2^- and d_3^- also indicated by arrows and are shown in the figure. Once all the goal constraints have been plotted on the graph, the feasible region is represented by the shaded region *OABC*.

Step 2. This step analyse each goal in the objective function to search the optimal solution in the feasible region. The first goal is completely achieved by the line *EF* in the reduced feasible region *EBF*. Then, we should try to achieve the sales goal of product *A* (since the differential weight is twice). Since at the second priority level, the sales goal constraint of product *A* has more weight (twice) than the product B, the manager would like to achieve first the sales goal of product *A* which can be completely achieved, and shown by the line *EB* in the feasible region *EBF*. Also the second priority given to product *B* is also completely achieved and shown by the line *FB* in the feasible region *EBF*. Thus, the first two goals are completely achieved at point *B* in the feasible region *EBF*. The third priority is given to the overtime operation which cannot be achieved at the expense of first two goals. Now, the solution to this problem becomes

$$x_1 = 240, \quad x_2 = 300, \quad d_1^- = 0, \quad d_2^- = 0, \quad d_3^- = 0, \quad d_1^+ = 140$$

Remark : We can also observe that if the second priority is given to the overtime operation and the third priority to sales goal constraint, then the solution will be on the line *EF*. Since the product *A* has more weight (twice) than the product *B*, the solution will be at the point *E* which satisfies the higher weight product *A* at the expense of lower weight product *B*. Then, the solution values become :

$$x_1 = 240, \quad x_2 = 160, \quad d_1^- = 0, \quad d_2^- = 0, \quad d_3^- = 140, \quad d_1^+ = 0$$

But, in goal programming the purpose is to minimize the unattained portion of goal as much as possible. This is achieved by minimizing the deviational variables through the use of preemptive priority factors and differential weights, because there is no profit maximization or cost minimization in the objective function (the preemptive factors and differential weights represent c_j values).

Also, it should be remembered that preemptive priority factors are ordinal weights and they are not commensurable. Thus, z_j or $(z_j - c_j)$ cannot be expressed by a single row as in linear programming. Rather the simplex criterion becomes an $(m \times n)$ size matrix, where m represents the number of preemptive priority factors and n is the number of variables (both decision and deviational).

And since in the simplex criterion, $z_j - c_j$ is expressed by a matrix rather than a row, a new procedure must be devised for identifying the key column. Again, since $P_1 >> P_{j+1}$, the selection procedure of the key column must be initiated from P_1 and move gradually to the lower priority levels.

13.5-2. Simplex Method Applied to GP Problems

The standard simplex method can be used to solve goal programming problems. Table 13.1 showed the solution of a *single-goal model* solved *via* the simplex method. Table 13.2 showed the solution of a *multigoal model without priorities*. The simplex method can also be used to solve other GP models. As a matter of fact, any goal programming problem can be solved with the standard algorithm. As indicated in the discussion of the modified production example, models (13.22)– (13.27), this is accomplished by assigning values to the priority coefficients in the objective function. These values should be assigned so that the values reflect the same order of relationship as the priorities. For example, for a three-priority model, P_1 could be assigned a value of 100, P_2 a value of 10, and P_3 a value of 1.

For the revised production problem, the objective function would appear as follows :

$$\textit{Minimize}: \quad z = 100d_2^- + 10d_3^- + 1d_1^- \tag{13.51}$$

For the extended model, (13.33) – (13.39), the objective function would be

$$\textit{Minimize}: \quad z = 100d_6^+ + (1)(10)d_2^- + 2(10)d_3^- + 1d_1^- \tag{13.52}$$

For the previous graphical problem, model (13.45) – (13.50), the objective function would be

$$\textit{Minimize}: \quad z = 1000d_1^+ + 1000d_2^+ + (30)(10)d_3^- + 40(10)d_4^- + d_5^- \tag{13.53}$$

In the latter case, P_1 was assigned a value of 1000 and P_2 a value of 10. Had we assigned P_1 a value of 100, the second set of goals ($20P_2d_3^-$ and $40P_2d_4^-$) would have taken precedence over the first priority goal since the weights of 30 and 40 are employed in the second-level goals.

Then the advantage of expressing the objective function in this manner is that any GP problem can be solved without having a GP algorithm. However, if a GP algorithm is available it should be employed, since the table-to-table output from the algorithm provides some useful information. Use of a GP algorithm can also be justified by the fact that additional effort is required to interpret the output of a GP model solved *via* the standard simpelx method.

13.5–3. The GP Algorithm : Extended Simplex Method

We have described in detail the steps and procedures of simpelx method. In this section, we will demonstrate how the algorithm can be modified to solve a goal programming model. As in the case for most algorithms, the GP algorithm is intended for use on the computers. However, if the student is to gain any appreciation or feeling for the method, several problems must be calculated by hand. As the case for the simplex method, output from a computerized version of the algorithm can be.

Following steps are employed in the GP algorithm.

Step 1. Construct the initial, modified simplex table. This table will include a $(z_j - c_j)$ row for each priority level. Set $k = 1$, where k is a pointer which represents the $(z_j - c_j)$ row associated with P_k priority. Go to *step 2.*

Step 2. Check for optimality by examining the $\mathbf{X_B}$ (solution) value in the $(z_j - c_j)$ row for the P_k priority. If a zero exists, then the P_k priority goal has been met. Therefore, go to *step 6.* If a zero value does not exist, proceed to *step 3.*

Step 3. Determine the new entering variable by examining $(z_j - c_j)$ row for the P_k priority. Examine each *positive* coefficient in the P_k row. Identify the largest positive coefficient for which there are no negative coefficients at a higher priority in the same column. The variable in the column associated with the largest positive coefficient is the incoming variable. If a tie exists in the values of the coefficients that determine the entering variable, it can be broken arbitrarily. If no positive coefficient exists in the P_k row that meets the above conditions, then go to *step 6;* otherwise, go to *step 4.*

Step 4. Determine the departing variable by using the standard procedure from the simplex method.

Step 5. Develop the new table. The standard procedure of the simplex method is used to update the coefficients in the body of the table. The new $(z_j - c_j)$ rows are computed in the same manner used in developing the initial table. Go to *Step 2.*

Step 6. Evaluate the next-lowest priority level by setting $k = k + 1$. If k exceeds K, where K is the total number of priority levels, then stop; the solution is optimal. If $k \le K$, then $(z_j - c_j)$ row for the P_k priority level must be examined. Therefore, go to *step 2.*

Demonstration : To demonstrate the algorithm, consider again the problem which we solved graphically. The GP model for the problem was formulated as follows [refer to equations (13.94) – (13.99)] :

$$\text{Minimize}: \quad z = P_1 d_1^+ + P_1 d_2^+ + 30P_2 d_3^- + 40P_2 d_4^- + P_3 d_5^-$$

$$\begin{aligned}
\text{Subject to}: \quad & 2x_1 + 4x_2 + d_1^- - d_1^+ = 80 \\
& 3x_1 + 3x_2 + d_2^- - d_2^+ = 80 \\
& x_1 + d_3^- - d_3^+ = 10 \\
& x_2 + d_4^- - d_4^+ = 10 \\
& 30x_1 + 40x_2 + d_5^- - d_5^+ = 1200 \\
& x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^+, d_4^-, d_4^+, d_5^-, d_5^+ \ge 0.
\end{aligned}$$

Step 1 : Establish the initial table. The procedure for constructing the initial table is identical to that of simplex method, with the exception of the last row(s) of the table. We begin by determining if an initial feasible solution exists within the coefficients of the constraints. If a d^- variable is associated with each goal constraint, it will be the basic variable for that constraint. If structural constraints exist, the slack variables will be in the initial basis; if slack variables do not exist, artificial variables are added. For the current problem, d_1^- through d_5^- form the basis for the first table.

The initial table for the problem is presented in Table 13.3. The coefficients in the c_j row, the C_B column, and in the body of the table are transferred into the table in the same manner used in the simplex method. The difference between this table and any initial simplex table is the $(z_j - c_j)$ row.

Remember that we solved minimization LP problems with the simplex method by multiplying each coefficient in the objective function by – 1. Since every GP problem is a minimization problem, logic would tell us that we must multiply the objective function by – 1 before employing the simplex method.

The multiple $(z_j - c_j)$ rows are computed in the same manner as would be used in the simplex method. The difference is the tabular representation of the results. Note that z_j is computed by multiplying the value in the jth column by the values in the C_B column and summing the component parts. For example, for the x_1 column

$$z_1 = (2)(0) + (3)(0) + (1)(30P_2) + (0)(40P_2) + (30)(P_3)$$

The value of c_1 (from the c_j row) associated with x_1 is zero. Therefore,

$$z_1 - c_1 = 0 + 0 + 30P_2 + 0 + 30P_3 - 0$$

Rather than transfer this into a single cell in a single $(z_1 - c_1)$ row, we can break it into parts where the parts are associated with the priority levels. In this case we have

$$\begin{array}{ll} 30 & P_3 \\ 30 & P_2 \\ 0 & P_1 \end{array}$$

By performing this procedure for each column, we develop the multiple $(z_j - c_j)$ rows and expressing the $(z_j - c_j)$ rows in this manner, we can in later steps in the algorithm seek to satisfy each goal in priority order.

Table 13.3 : Initial Table

	$c_j \to$		0	0	0	P_1	0	P_1	$30P_2$	0	$40P_2$	0	P_3	0	
Basic Var.	C_B	Right-hand Side	X_1	X_2	d_1^-	d_1^+	d_2^-	d_2^+	d_3^-	d_3^+	d_4^-	d_4^+	d_5^-	d_5^+	Min. Ratio
d_1^-	0	80	2	4	1	–1	0	0	0	0	0	0	0	0	80/4
d_2^-	0	80	3	3	0	0	1	–1	0	0	0	0	0	0	80/3
d_3^-	$30P_2$	10	1	0	0	0	0	0	1	–1	0	0	0	0	—
d_4^-	$40P_2$	10	0	$\boxed{1}$	0	0	0	0	0	0	1	–1	0	0	10/1 ←
d_5^-	P_3	1200	30	40	0	0	0	0	0	0	0	0	1	–1	1200/40
$z_j - c_j$	P_3	1200	30	40	0	0	0	0	0	0	0	0	0	–1	
	P_2	700	30	40	0	0	0	0	0	–30	0	–40	0	0	$\leftarrow \Delta_j$
	P_1	0	0	0	0	–1	0	–1	0	0	0	0	0	0	

We begin examining the initial table by checking $(z_j - c_j)$ row associated with the P_1 priority level (*i.e.*, the index pointer, k, is set to 1).

Step 2. Check for optimality. Remember that, in the simplex method, optimality exists when the value of every coefficient in the $(z_j - c_j)$ row is zero or positive. For the GP algorithm, two checks can be used to test for optimality : (1) Is there a zero in the *right-hand side (solution)* column (RHS column) in the P_k row, or (2) Are all values in the P_k row zero or positive ? A zero in the RHS column of a given row indicates that the associated goal has been achieved. For our problem, the RHS value in the P_1 row is zero. Therefore, the P_1 goal has been met. A quick check of all values in the P_1 row verifies this.

In the RHS column, the 700 associated with the P_2 row and the 1200 associated with the P_3 row indicate the extent to the goals at these priority levels have not been satisfied. By examining the solution values in the table we can explain these achievement levels as well as that of P_1.

The solution at this step is. $d_1^- = 80$, $d_2^- = 80$, $d_3^- = 10$, $d_4^- = 10$, and $d_5^- = 1200$; all other variables are zero since they are nonbasic variables. The top-priority goal (P_1) is to avoid overtime in either department; since d_1^+ and d_2^+ are zero, we have achieved this goal. The second priority goal (P_2) is to meet production goals of 10 units of x_1 and 10 units of x_2. We missed both of these goals by 10, since d_3^- and d_4^- are each 10. The weighted value of d_3^- is 30 and the weighted value of d_4^- is 40 (at the P_2 level). Therefore, 700 is the *weighted* value of our under-achievement of the total production goal. The third priority goal is to achieve a profit of Rs 1200 $(d_5^- = 1200)$.

Since the top-priority goal is met, the GP algorithm indicates that we should set the k index pointer to 2 and examine the $(z_j - c_j)$ row associated with P_2.

From the above analysis, we already know that P_2 is not achieved. A check of the values of the coefficients in the $(z_j - c_j)$ row verifies this since a + 30 exists in the X_1 column and a + 40 exists in the X_2-column. Since no negative values exist in columns X_1 or X_2 at the P_1 level, we can proceed to *step 3.*

Step 3 : Determine the new entering variable. The largest positive coefficient in the $(z_j - c_j)$ row asociated with P_2 is 40. Therefore, the *incoming variable* is x_2.

Step 4 : Determine the departing variable. To determine the departing variable we use the standard simplex procedure of dividing the positive coefficients in the x_2 column into the values in the RHS column and selecting the smallest ratio. In this case we have $80/4 = 20$; $80/3 = 26.66$; $10/0 = \infty$; $10/1 = 10$, and $1{,}200/40 = 30$. The smallest ratio is associated with d_4^-. Therefore, d_4^- is the *outgoing variable.*

Step 5 : Develop the new table. The new table is developed by employing the up-dating procedure used in the simplex method : the incoming row is computed by dividing the outgoing row by the pivot element; all remaining rows are updated by use of the new incoming row and the old row in the table. The new $(z_j - c_j)$ rows are computed using the procedure employed in the initial table. Table 13.4 is the second table (Table II) for the problem.

Table 13.4 : Second Table

		$c_j \rightarrow$	0	0	0	P_1	0	P_1	$30P_2$	0	$40P_2$	0	P_3	0
Basic Var.	C_B	Right-hand Side X_B	X_1	X_2	d_1^-	d_1^+	d_2^-	d_2^+	d_3^-	d_3^+	d_4^-	d_4^+	d_5^-	d_5^+
d_1^-	0	40	2	0	1	−1	0	0	0	0	−4	4	0	0
d_2^-	0	50	3	0	0	0	1	−1	0	0	−3	3	0	0
d_3^-	$30P_2$	10	$\boxed{1}$	0	0	0	0	0	1	−1	0	0	0	0
x_2	0	10	0	1	0	0	0	0	0	0	1	−1	0	0
d_5^-	P_3	800	30	0	0	0	0	0	0	0	−40	40	1	−1
$z_j - c_j$	P_3	800	30	0	0	0	0	0	0	0	−40	40	0	−1
	P_2	300	30	0	0	0	0	0	0	−30	−40	0	0	0
	P_1	0	0	0	0	−1	0	−1	0	0	0	0	0	0

Examination of Table 13.4 indicates that the second priority goal (P_2) has not been attained, since the RHS value in the $(z_j - c_j)$ row for P_2 is 300 and a positive value exists in the x_1 column. However, the solution at this point is a better solution since the weighted deviation from the P_2 goal have been reduced from 700 to 300. This reduction occurs because d_4^- has been driven to zero and d_4^- is weighted by 40 in the objective function.

The solution values for Table 13.4 are $d_1^- = 40$, $d_2^- = 50$, $d_3^- = 10$, $x_2 = 10$, and $d_5^- = 800$. These indicate that 40 hours of unused capacity remains in department 1; 50 hours of unused capacity remains in department 2; the production goal of x_1 is still under-achieved by 10 units; 10 units of x_2 are produced; and the profit goal is still under-achieved by Rs. 800. From this information and from the solution in the initial table, we can conclude that the production of 10 units of x_2 required 40 hours in department 1 and 30 hours in department 2, and resulted in Rs. 400 of profit (which moves Rs. 400 closer to achieving the profit goal of Rs. 1200).

Since we were not able to fully achieve the second priority goal (P_2), we must check to determine if we can continue the table process. (This action sends us back to *step 2* in the GP algorithm). The positive 30 that exists under the x_1 column in the $(z_j - c_j)$ row for P_2 indicates that x_1 is a condidate for entering the basis. Since a negative number does not exist under the x_1 column in the $(z_j - c_j)$ row of P_1, we can bring x_1 into the basis without violating the existing goal achievement at the P_1 level.

A quick check of the min. ratio of the values in the x_1 column and the right-hand side values ($^{40}/_{2}$, $^{50}/_{3}$, $^{10}/_{1}$, $^{10}/_{0}$, $^{800}/_{30}$) indicates that d_3^- should be removed from the basis.

If we compute the new table that results from removing d_3^- and bringing x_1 into the basis it will be discovered that the first two priority goals are fully achieved. However, this will not be the best solution, *i.e.*, the optimality criteria will not be met. Thus, we should continue the pivoting procedure.

Step 6. Evaluate the next lowest priority goal(s). To evaluate the next priority goal, (third in this case), we add one to the pointer index, k, (note that k was equal to 2, which pointed us to the P_2 priority level) and evaluate the new P_k row of the current table. This would indicate what we already know, that P_3 should be examined for optimality. If we perform this step, a new table (fourth, in this case) will result. As was the case for the third table, this fourth table will not provide the best solution. However, the fifth table, which would result from one additional pivot, will provide the best solution. This table is shown in Table 13.5.

Table 13.5 : Optimal Table

		$c_j \rightarrow$	0	0	0	P_1	0	P_1	$30P_2$	0	$40P_2$	0	P_3	0
Basic Var.	C_B	Right-hand Side X_B	X_1	X_2	d_1^-	d_1^+	d_2^-	d_2^+	d_3^-	d_4^+	d_4^-	d_4^+	d_5^-	d_5^+
d_4^+	0	3.33	0	0	.5	−.5	−.33	.33	0	0	−1	1	0	0
d_3^+	0	3.33	0	0	−.5	.5	.66	−.66	−1	1	0	0	0	0
x_1	0	13.33	1	0	−.5	.5	.66	−.66	0	0	0	0	0	0
x_2	0	13.33	0	1	.5	.5	−.33	.33	0	0	0	0	0	0
d_5^-	P_3	266.66	0	0	5	.5	−6.6	6.6	0	0	0	0	1	−1
$z_j - c_j$	P_3	266.66	0	0	−5	5	−6.6	6.6	0	0	0	0	0	−1
	P_2	0	0	0	0	0	0	0	−30	0	−40	0	0	0
	P_1	0	0	0	−1	−1	0	−1	0	0	0	0	0	0

Interpreting this table, we note that we are now Rs. 266.66 short of the profit goal of Rs. 1200— but we are still under-achieving the goal. We have used all the available production capacity ($d_1^- = 0$, $d_2^- = 0$). Therefore, we cannot, logically improve our profit goal without going to overtime (which would violate the P_1 goal). A check of the $(z_j - c_j)$ row for P_3 confirms that this is true—a positive value 6.6 exists under the d_2^+ column in the $(z_j - c_j)$ row of P_3 but a negative value − 1 exists in the same column in the $(z_j - c_j)$ row of P_1 ; likewise a positive value (5) exists under d_1^+ column in $(z_j - c_j)$ row of P_3 but a negative value (− 1) exists in the same column in the $(z_j - c_j)$ row of P_1. Therefore, we must conclude that this is the best solution for the defined problem. This solution is identical to graphical solution :

$$x_1 = 13.33,\ x_2 = 13.33, d_5^- = 266{\cdot}66,\ d_3^+ = 3{\cdot}33,$$
$$d_4^+ = 3{\cdot}33,\ d_1^- = d_2^- = d_3^- = d_4^- = d_1^+ = d_2^+ = d_5^+ = 0.$$

13.6. SPLECIAL PROBLEMS IN GP

A number of special problems can occur in the GP–simplex solution process, such as ties for the departing variable, infeasible solutions, alternate optimal solutions, and negative right-hand sidé values. Since some of these can have an impact on the solution, the following rules are applied in order :

Tie for Departing Variable. In constructing a GP-simplex table, the departing variable is determined by the smallest nonnegative ratio that results when the coefficients in the incoming column are divided into the *right-hand side*. If two or more rows have the same ratio, the tie may be broken by selecting the row with the highest associated priority level (located in the C_B column). In some cases, the variable being considered for removal will not have an associated priority level. In such cases, ties may be broken arbitrarily. Although theory would indicate that this could cause cycling because of degeneracy. But this has not occurred in practice.

Infeasible Solution. As indicated earlier, infeasible solutions, occur in linear programming, but not in goal programming. However, an *unexpected* solution may occur. One should not expect that, when a given priority goal is not achieved, every lower priority goal will likewise not be achieved. It is possible to have a solution in which the top priority goal is not achieved but lower priority goals are met. This generally occurs when absolute objectives (goals) exist and limits (bounds) are placed on resources.

Alternate Optimal Solutions. Alternate optimal can occur in goal programming just as in linear programming. The existence of alternate optimal solutions is indicated by an entire column of zeros in the $(z_j - c_j)$ rows for a non-basic variable and the existence of at least one positive a_{ij} element in the corresponding column. The alternate solution is determined by computing the new table, using the standard update procedure.

Negative Right-hand Side Values. The GP–simplex table procedure generally cannot accept negative right-hand-side values, although some computer codes have special procedures to accept such conditions. One can avoid negative right-hand side values simply by multiplying the entire goal constraint by – 1. The modification should be made after the deviational variables have been added. For example, the goal constraint

$$3x_1 - 4x_2 - 5x_3 + d_1^- - d_1^+ = -50$$

would be modified to

$$-3x_1 + 4x_2 + 5x_3 - d_1^- + d_1^+ = 50.$$

The appropriate variable to appear in the objective function is determined from analysis of the original constraint, not the new constraint. That is, if the goal were to achieve or exceed – 50, then d_1^- would be placed in the objective function; if the goal were to under-achieve – 50 , then the d_1^+ would appear in the objective function. (In some computer codes the deviational variables are added automatically which means they would be added to the modified constraint. Therefore, it may be necessary to reverse the variables when they are analyzed in the output.

Q. 1. Explain the differences in solving a linear versus goal programming problem by the simplex method.

2. Give the difference between linear programming and goal programming. **[IGNOU (MCA II) 2000]**

13.7. OTHER MULTICRITERIA METHODS

From examples, it is clear that the decision maker may wish to experiment with several priority arrangements before accepting a given solution. Such an analysis is difficult within goal programming (unless one is willing to make several runs of a model), but is provided by other multi-criteria models, specially interactive multi-criteria models. However, most of the interactive methods, require inputs beyond those required for the basic goal programming method.

The advantage of these, as well as other multi-criterion methods, is that the decision maker can examine alternatives rather than simply accept a given solution resulting from a given set of priority levels. The disadvantage of most of these methods is that large problems cannot be solved very efficiently and an interactive computer system is required. With some of the recent developments in algorithm solution structures and recent advances in computer technology, these methods should become more attractive in the future.

Decision makers in an organization often are faced with problems that have multiple (and often conflicting) goals. In this chapter, we have discussed that if such problems can be described by linear goal equations (constraints) and if the goals can be prioritized (ranked) in terms of importance, then goal programming is a viable multi-criteria programming technique for generating solutions to the problems.

Obviously, goal programming does not provide answers to all multi-criteria problems. The technique will provide solutions for problems where goals are to be met in a preferential order. However, research is needed in the areas of sensitivity analysis and solutions that will allow a small deviation from a higher priority goal in order to achieve a substantial reduction in a lower priority goal or goals. Research into under-standing duality theory's relationship to goal programming should also broaden the use of the technique. If the technique is to be fully functional, it should also provide integer solution values for problems where this is a strict requirement.

Illustrative Example

Example 9. *Use modified simplex method to solve Example 8. The complete goal programming formulation is again reproduced as :*

$$\text{Minimize}: \quad z = P_1 d_1^- + 2P_2 d_2^- + P_2 d_3^- + P_3 d_1^+,$$

$$\text{Subject to}: \quad x_1 + x_2 + d_1^- - d_1^+ = 400$$

$$x_1 + d_2^- = 240$$

$$x_2 + d_3^- = 300$$

and $$x_1, x_2, d_1^-, d_2^-, d_3^-, d_1^+ \geq 0.$$

Solution. The initial simpiex table for this problem is given in the table. The first three rows are shown in the same way as for the linear programming problem. Beneath the thick line, each row stands for the priority goal level. The highest priority goal P_1 is put the bottom and the lower ranked goals P_2 and P_3 are placed above P_1 in ascending order. since the initial solution is at the point of origin, there is no production and all resources are idle. Thus, all negative deviational variables will appear in the initial solution. Now construct the initial simplex table, and we get the initial solution as

$$d_1^- = 400\,, d_2^- = 240\,, d_3^- = 300$$

The simplex criterion $z_j - c_j$ is 3×6 matrix (given below the thick line) consisting of 3 priority levels and 6 variables (2 decision + 4 deviational). Here the c_j values represent the priority factors assigned to deviational variables and the z_j values are calculated as usual as explained earlier. To obtain the values of $z_j - c_j$, the z_j values are to be computed orally and c_j values are then subtracted from the z_j values.

Once the initial solution is obtained, we can apply the standard simplex method for the *minimization* case with few modifications. First, to select the *key column,* we must select the largest positive element in the highest priority level P_1. In this example, the largest positive element 1 in the P_1 row occurs at two places. To remove tie, check the next lower priority level and select the key column that has a greater value at the lower priority level. So, we select the variable x_1 as our key column. Next, in choosing the *key row,* we divide the values of $\boldsymbol{b}$ (= $\mathbf{X_B}$) by the corresponding coefficients in the key column (ignoring zero and negative coefficients) and select that row for which this ratio is minimum. In this example, we select that row for which this ratio is minimum. In this example, we select row 2 as the *key row* as shown in Table 13.6. The inter-section between the key row and key column is called the *key element* and is squared in Table 13.6.

Table 13.6

		$c_j \rightarrow$	0	0	P_1	$2P_2$	P_2	P_3	
Ver. in Basic	C_B	X_B (Resources)	X_1	X_2	d_1^-	d_2^-	d_3^-	d_1^+	Min. Ratio (X_3/X_1)
d_1^-	P_1	400	1	1	1	0	0	−1	400
d_2^-	$2P_2$	240	$\boxed{1}$	0	0	1	0	0	240 ← key row
d_3^-	P_2	300	0	1	0	0	1	0	—
	P_3	0	0	0	0	0	0	−1	
	P_2	780	2	1	0	0	0	0	$\Delta_j = z_j - c_j$
	P_1	400	1↑	1	0	0↓	0	−1	

Following is the standard simplex method, the first table is revised to obtain the second improved table as shown in Table 13.7. In Table 13.7, we notice that d_2^- has been replaced by x_1. Proceeding in the like manner, we find variable x_2 becomes the key column and row 1 as the key row. Therefore, the key element 1 squared in Table 13.7 suggests that the variable x_2 will enter into the solution and d_1^- will be driven out.

Table 13.7

		$c_j \rightarrow$	0	0	P_1	$2P_2$	P_2	P_3	
Basic Var.	C_B	X_B	X_1	X_2	d_1^-	d_2^-	d_3^-	d_1^+	Min. Ratio (X_B/X_2)
d_1^-	P_1	160	0	$\boxed{1}$	1	−1	0	−1	160 ← key row
x_1	0	240	1	0	0	1	0	0	—
d_3^-	P_2	300	0	1	0	0	1	0	300
	P_3	0	0	0	0	0	0	−1	
	P_2	300	0	1	0	−2	0	0	Δ_j
	P_1	160	0	1↓	0↓	−1	0	−1	

Table 13.8

		$c_j \rightarrow$	0	0	P_1	$2P_2$	P_2	P_3	
Basic Var.	C_B	X_B	X_1	X_2	d_1^-	d_2^-	d_3^-	d_1^+	Min. Ratio (X_B/d_1^+)
x_2	0	160	0	1	1	−1	0	−1	—
x_1	0	240	1	0	0	1	0	0	—
d_3^-	P_2	140	0	0	−1	1	1	$\boxed{1}$	140←
	P_3	0	0	0	0	0	0	−1	
	P_2	140	0	0	−1	−1	0	1	Δ_j
	P_1	0	0	0	−1	0	0↓	0↑	

The revised Table 13.10 suggest that all the elements are either zero or negative in the P_2 row. Thus, second goal is also fulfilled. In an attempt to satisfy the third goal, we notice that there are two positive elements, *viz.* 1 and 1 *in the P_3 row.* However, these positive elements will not be considered, because there is already a negative element at a higher priority level. Since element 1 in P_3 row is above the elements – 2 and – 1 in the P_2 row, the rule is that if there is a positive element at a lower priority level in $z_j - c_j$, the variable in that column cannot be introduced into the solution as long as there is a negative element at a higher priority level.

Table 13.9

		$c_j \rightarrow$	0	0	P_1	$2P_2$	P_2	P_3
Basic Var.	C_B	X_B	X_1	X_2	d_1^-	d_2^-	d_3^-	d_1^+
x_2	0	300	0	1	0	0	1	0
x_1	0	240	1	0	0	1	0	0
d_1^+	P_3	140	0	0	−1	1	1	1
	P_3	140	0	0	−1	1	1	0
	P_2	0	0	0	0	−2	−1	0
	P_1	0	0	0	−1	0	0	0

Table 13.9 represents the optimal solution and the solution is given by

$$x_1 = 240, x_2 = 300, d_1^- = d_2^- = d_3^- = 0, d_1^+ = 140$$

In this example, *priority P_1 and P_2 corresponding to goal* 1 *and* 2 *are completely achieved, and P_3 corresponding to goal* 3 *is not achieved.*

Example 10. *A textile company produces two types of materials, A and B. The material A is produced according to direct orders from furniture manufactures. The material B, is distributed to retail fabric stores. The average production rates for the material A and B are identical to 1000 metres/hour. By running two shifts the operational capacity of the plant is 80 hours per week.*

The marketing department reports that the maximum estimated sales for the following week is 70,000 metres of material A and 45,000 metres of material B. According to the accounting department the profit from a metre of material A is Rs. 2.50 and from a metre of material B is Rs. 1.50.

The management of the company decides that a stable employment level is a primary goal for the firm. Therefore, whenever there is demand exceeding normal production capacity, management simply expands production capacity by providing overtime. However, management feels that overtime operation of the plant of more than 10 hours per week should be avoided because of the accelerating costs. The management has the following goals in order of their importance.

1. Avoid any underutilization of production capacity.

2. Limit the overtime allowed for plant operation to 10 hours per week.

3. Achieve the sales target of 70,000 metres of material A and 45,000 metres of material B.

4. Minimize overtime operation of the plant as much as possible.

Formulate this problem as a goal programming problem and derive its solution by simplex method.

Formulation :

Production Hours Constraint. Regular production hours available to the firm are limited to 80 hours. If necessary, management may allow some extra hours of work (overtime). So the production hours constraint could be set up as follows :

$$x_1 + x_2 + d_1^- - d_1^+ = 80$$

where x_1 = number of hours used for producing the material A.

x_2 = number of hours used for producing the material B.

d_1^- = amount of under-utilization of available normal production hours.

d_1^+ = amount of extra hours (overtime) beyond normal production hours allowed by the management.

Sales Constraint. Since no more than the stated sales target of 70,000 metres of material A and 45,000 metres of material B can be sold in the market and as x_1 and x_2 denote to produce 1,000 metres per hour of material A and B respectively, the sales constraint will be written as

$$x_1 + d_2^- = 70 \quad \text{and} \quad x_2 + d_3^- = 45$$

where d_2^- = amount of under-achievement of sales target of material A.

d_3^- = amount of under-achievement of sales target of material B.

Overtime Constraint. The management would like to limit overtime d_1^+ to 10 hours or less. Then the overtime constraint may be introduced as follows :

$$d_1^+ + d_4^- - d_4^+ = 10$$

where d_1^+ = actual overtime operation of the plant.

d_4^- = amount of under-utilized hours between the actual overtime allowed and 10 hours of overtime.

d_4^+ = amount of overtime in excess of the allowed 10 hours.

Both negative and positive deviations are included from the 10 hours of overtime allowed because the actual overtime can be less than, equal to, or even more than 10 hours. To achieve this goal, the management can minimize the deviational variable d_4^+.

Alternatively, the overtime constraint can be incorporated directly into the regular production hours constraint. This can be obtained by substituting the value of d_1^+ from the overtime constraint into the production hours constraint as given below :

$$x_1 + x_2 + d_1^- - d_1^+ = 80$$

$$x_1 + x_2 + d_1^- - (10 - d_4^- + d_4^+) = 80$$

$$x_1 + x_2 + d_4^- - d_4^+ = 90$$

In this constraint, the question of under-utilization (d_1^-) will not arise when we take the constraint for overtime operation. This is also true because $d_1^- \times d_1^+ = 0$, and since $d_1^+ > 0$ so $d_1^- = 0$.

Objective Function. With priority-ranked goals, one objective function have to be formulated for each goal in the goal programming model. The management have ranked goals as per their importance as follows :

P_1 : *To avoid any under-utilization of production capacity.*

P_2 : *To limit overtime operation of the plant to 10 hours.*

P_3 : *To achieve the sales goal of 70,000 metres of material A and 45,000 metres of material B.*

P_4 : *To minimize the overtime operation of the plant as much as possible.*

Since sales goals for materials are considered equally important, the profit contribution ratio (5 : 3) between these two will be considered as differential weights.

The complete goal programming model for the problem can be formulated as follows :

Minimize : $z = P_1 d_1^- + P_2 d_4^+ + 5P_3 d_2^- + 3P_3 d_3^- + P_4 d_1^+$,

Subject to : $x_1 + x_2 + d_1^- - d_1^+ = 80$

$$x_1 + d_2^- = 70$$

$$x_2 + d_3^- = 45$$

$$x_1 + x_2 + d_4^- - d_4^+ = 90$$

and $\quad x_1, x_2, d_1^-, d_2^-, d_3^-, d_4^-, d_1^+, d_4^+ \geq 0.$

Solution. The solution to this problem by simplex method is explained as follows :

The initial simplex table is presented in Table 13.10. The first four rows of the table are set up in the same way as for a L.P. problem, with the coefficients of the associated variables placed in the appropriated entries. Below the thick line which separates the constraints from the objective function, there are four rows and each row stands for a priority goal level.

Table 13.10

		$c_j \rightarrow$	0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2	
Basic Variables	C_B	Resources X_B	X_1	X_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_4^+	Minimum ratio X_B/X_1
d_1^-	P_1	80	1	1	1	0	0	0	−1	0	80
d_2^-	$5P_3$	70	$\boxed{1}$	0	0	1	0	0	0	0	70 ←
d_3^-	$3P_3$	45	0	1	0	0	1	0	0	0	—
d_4^-	0	90	1	1	0	0	0	1	0	−1	90
	P_4	0	0	0	0	0	0	0	−1	0	
	P_3	485	5	3	0	0	0	0	0	0	$\leftarrow \Delta_j$
	P_2	0	0	0	0	0	0	0	0	−1	
	P_1	80	1↑	1	0	0↓	0	0	−1	0	

Note. Note that the optimal criterion $z_j - c_j$ is a 4×8 matrix because there are four priority factors and eight variables (two decision variables and six deviational variables) in the model. Preemptive priority goals are written in basic variables column X_1 below the thick line from the lowest at the top to the highest at the bottom.

It should be apparent that the selection of the key column is based on the per unit contribution rate of each variable in achieving the goals. When the first goal is completely attained, then the key column selection criterion moves on to the second goal, and so on. This is why the preemptive priority factors are listed from the lowest to the highest so that the key column can be easily identified at the bottom of the Table 13.10. To make the simplex table relatively simple, the z_j matrix is omitted.

In goal programming, the z_j values ($P_4 = 0$, $P_3 = 485$, $P_2 = 0$ and $P_1 = 80$) in the resources column represents the unattained portion of each goal.

The key column would be determined by selecting the largest positive element in $z_j - c_j$ row at the P_1 level, as there exists an unattained portion of this highest goal ($P_1 = 80$). There are two identical positive $z_j - c_j$ values in the x_1 and x_2 columns. In order to break this tie, we check the subsequently next lower priority levels. Since at priority 3, the largest element is 5 in that row, x_1 becomes the *key column*. The values of $\Delta_j = z_j - c_j$ are computed as follows :

Δ_j = (elements in C_B column $\times$ corresponding elements in X_j columns) − C_j (priority factors assigned to deviational variables)

For example,

$\Delta_1 = P_1 \times 1 + 5P_3 \times 1 + 3P_3 \times 0 + 0 \times 1 = P_1 + 5P_3$ for column X_1 Similarly,

$\Delta_2 = P_1 + 3P_3$ for column X_2 $\quad \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 0$ because d_1^-, d_2^-, d_3^- and d_4^- are the basic variables.

$\Delta_7 = -P_1 - P_4$ for d_1^+ column. $\quad \Delta_8 = -P_2$ for d_4^+ column.

Since P_1, P_2, P_3 and P_4 are not commensurable, we must list their coefficeints separately in their rows in the simplex criterion $\Delta_j = (z_j - c_j)$ as shown in Table 13.10.

The key row is determined by selecting the minimum positive or zero value when values in the resources column are divided by the coefficients in the key column. In this problem, the *key row* refers to d_2^- row as shown in Table 13.10. If there exists a tie, then select the row that has the variable with the highest priority factor. Coefficient lying at the intersection of *key row*, and *key column* is called *key element* and in Table 13.10, the key element 1 is squared.

By utilizing the usual simplex procedure, Table 13.11 is revised to obtain Table 13-12 as shown below :

Table 13.11

Basic Variables	C_B	Resources X_B	X_1	X_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_4^+	Min. ratio X_B/X_1
		$c_j \rightarrow$	0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2	
d_1^-	P_1	10	0	$\boxed{1}$	1	−1	0	0	−1	0	10←
x_1	0	70	1	0	0	1	0	0	0	0	—
d_3^-	$3P_3$	45	0	1	0	0	1	0	0	0	45
d_4^-	0	20	0	1	0	−1	0	1	0	−1	20
	P_4	0	0	0	0	0	0	0	−1	0	
	P_3	135	0	3	0	−5	0	0	0	0	←$(z_j - c_j)$
	P_2	0	0	0	0	0	0	0	0	−1	
	P_1	10	0	1↑	0↓	−1	0	0	−1	0	

Again, Table 13.11 does not give the optimal solution as the resources column (X_B) indicates unattained portion of goals. Proceeding in the usual manner, Table 13.11 suggest that an improved solution can be obtained if negative deviational variable d_1^- is driven out and decision variable x_2 enters into the solution. The new improved solution is shown in Table 13-12.

Table 13-12

Basic variables	C_B	Resources X_B	X_1	X_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_4^+	Minimum ratio X_B/X_1
		$c_j \rightarrow$	0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2	
x_2	0	10	0	1	1	−1	0	0	−1	0	—
x_1	0	70	1	0	0	1	0	0	0	0	—
d_3^-	$3P_3$	35	0	0	−1	1	1	0	1	0	35
d_4^-	0	10	0	0	−1	0	0	1	$\boxed{1}$	−1	10←
	P_4	0	0	0	0	0	0	0	−1	0	
	P_3	105	0	0	−3	−2	0	0	3	0	
	P_2	0	0	0	0	0	0	0	0	−1	$(z_j - c_j)$
	P_1	0	0	0	−1	0	0	0↓	0↑	0	

The solution in Table 13-12 indicates that production of 70,000 metres of material A and 10,000 metres of material B is sufficient to achieve the first, second and fourth goals and the value of $d_3^- = 35$ suggest that 35,000 metres of material is not achieved. It is also observed that all the elements in P_1 and P_2 row are either zero or negative which indicates that the first two priorities are achieved. Therefore, to improve the solution, the selection of key column is done at P_3 level. Since the only positive element (3) occurs at P_3 level which lies in d_1^+ column. Thus, d_1^+ enters into the solution and d_4^- is driven out as shown in Table 13.12. Finally, Table 13-13 presents the optimal solution. The solution is optimal in the sense that it enables the decision-maker to attain his goals as closely as possible within the given decision environment and priority structure.

Table 13-13 : Optimal Table

Basic Variables	C_B	Resources X_B	X_1	X_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_4^+
		$c_j \rightarrow$	0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2
x_2	0	20	0	1	0	−1	0	1	0	−1
x_1	0	70	1	0	0	1	0	0	0	0
d_3^-	$3P_3$	25	0	0	0	1	1	−1	0	1
d_1^+	P_4	10	0	0	−1	0	0	1	1	−1
	P_4	10	0	0	−1	0	0	1	0	−1
	P_3	75	0	0	0	−2	0	−3	0	3
	P_2	0	0	0	0	0	0	0	0	−1
	P_1	0	0	0	−1	0	0	0	0	0

The optimal solution is : $x_1 = 70$, $x_2 = 20$, $d_1^+ = 10$, $d_3^- = 25$. In this final simplex table, since the third goal is not completely attained, there is a positive value in $z_j - c_j$ at the P_3 level. This value is found as 3 in the d_4^+ column. Obviously, we can attain the third goal to a greater extent if we introduce d_4^+ in the solution. However, there is a negative value (– 1) at a higher priority level (P_2) in the same column. This implies that if we introduce d_4^+ into the solution we can improve achievement of the third goal at the expense of achieving the second goal which is not desirable. Similarly, there is single positive value 1 in d_4^- column at P_4 level and therefore fourth goal cannot be achieved at the expense of achieving the third goal. Therefore, the rule is that if there is a positive $z_j - c_j$ at a lower priority level, the variable in that column cannot be chosen as the entering variable as long as there is a negative $z_j - c_j$ at a higher priority level.

13.8. DEVELOPMENTS IN GOAL PROGRAMMING

Goal programming can be applied to a wide variety of decision problems invovling multiple objective functions. This is a powerful and flexible technique which is applicable only under certain specified assumptions and conditions. Applications are mostly limited to deterministic problems. In fact, the analysis of goal programming is limited to the identification of optimal solution. Various type of goal gprogramming models (linear, non-linear, linear integer, linear zero-one etc.) are avariable in the literature.

Goal programming is a powerful tool to tackle multiple and incompatible goals of any organization, some of which may be non-economic in nature. Goal programming can be applied to almost unlimited managerial decision areas. In the area of *marketing,* it can be applied to *media planning* and *product-mix* decisions. In *finance,* can be applied to portfolio selection, capital budgeting, and financial plannuing. In *production,* it can be applied to aggregate *production planning* and *scheduling*. In the *academic* field, it can be used in assigning *faculty teachings chedules* and for *university admissions planning*. In *HRD* area, it can be used for *manpower planning*. In *public systems* area, it can be applied to *transportation systems* and health-care planning. It may be pointed out that this list is not an exhaustive one but only indicative of the typical potential areas in which goal programming can be effectively applied.

Q. Identify the important areas where goal programming can be effectively used. **[IGNOU 2001]**

13.9. SUMMARY OF DEFINITIONS IN GP

Bounds. Omission of either the d^- or d^+ deviational variable in a goal constraint bounds the goal in the direction of the omission.

Deviational Variables. The d^- and d^+ variables incorporated into a goal equation—they effect the extent to which the goal is under-achieved (d^-), or exceeded (d^+).

Differential Weight. A numeric value (weight) assigned to goals (deviational variables) at the same priority level to reflect goal preference within the level.

Goal Equation. Used synonymously with the term *goal constraint. A goal equation is a goal expressed in mathematical equation form by inclusion of deviational variables.*

Goal Programming. A multi-criteria mathematical programming technique identified and studied by *A. Charmes* and *W.W. copper* in the early 1960s.

GP – simplex method. The modified simplex method (algorithm) used in solving a goal programming problem.

Multieriteria Mathematical Programming Techniques. Mathematical programming techniques that have the flexibility to incorporate more than one objective into the objective function. Goal programming is a multicriteria mathematical programming method; other methods exist.

Multiple Goals with Equal Priorities. A model in which deviational variables from more than one goal are incorporated into the objective function; all variables take on the same priority level.

Multiple Goals with Priorities. A model in which deviational variables from more than one goal are incorporated into the objective function; all variables are assigned a priority coefficient (P) which reflects the preferential ordering of the goals.

Multiple Goals with Priorities and Weights. A multiple-goal priority model in which differential weights are employed in one or more priority levels to differentiate goal preference within the level.

Priorities (P's). Coefficients assigned to goals (deviational variables) at the same priority level to reflect goal preference within the level.

Single Goal Model. A model in which the deivational variables associated with a single goal are incorporated into the objective function.

Structural Constraints. Environmental constraints that do not directly relate to the goals of the problem. Deviational variables are not incorporated into these constraints; therefore, the constraints are not incorporated into the objective function. These constraints place operational bonds (constraints) on a problem.

EXAMINATION PROBLEMS

1. A production manager faces the problem of job allocation between three of his teams. The processing rate of the three teams are 5, 6 and 8 units per hour respectively. The normal working hours for each team are 8 hours per day. The production manager has the following goals for the next day in order of priority :

   (i) The manager wants to avoid any under-achievement of production level, which is set at 180 units of product.

   (ii) Any overtime operation of team 2 beyond 2 hours and team 3 beyond 3 hours should be avoided.

   (iii) Minimize the sum of overtime.

   Formulate and solve the given problem as a goal programming problem.

2. *An office equipment manufacturer* produces two kinds of products, Chairs and Lamps. Production of either a Chair or a Lamp requires 1 hour of production capacity in the plant. The plant has a maximum production capacity of 10 hours per week. because of the limited sales capacity, the maximum number of Chairs and Lamps that can be sold are 6 and 8 per week, respectively. The gross margin from the sale of a Chair is Rs. 80 and Rs. 40 for a Lamp.
   The plant manager has set the following goals arranged in order of importance :
   (i) He wants to avoid any under-utilization of production capacity.

   (ii) He wants to sell as many Chairs and Lamps as possible. Since the gross margin from she sale of a Chair is set at twice the amount of profit from a Lamp, he has twice as much desire to achieve the sales goal for Chairs as for Lamps.

   (iii) He wants to minimize overtime operation of the plant as much as possible.

   Formulate this as a goal programming problem, so that the plant manager will make a decision that will achieve his goals as closely as possible. **[Meerut 2005]**

3. *ABC Furnitures* produce three products : tables, desks and chairs. All furniture is produced in the central plant. Production of desk requires 3 hours in the plant, a table takes 2 hours, and a chair requires only 1 hour. The regular plant capacity is 40 hours a week. According to marketing department, the maximum number of desks, tables and chairs that can be sold are 10, 10 and 12, respectively. The president of the firm has established the following goals according to their importance :

   (i) Avoid any under-utilization of production capacity, (ii) Meet the order of *XYZ Store* for seven desks and five chairs,

   (iii) Avoid the overtime operation of the plant beyond 10 hours, (iv) Achieve the sales goals of 10 desks, 10 tables, and 12 chainrs.

   (v) Minimize the overtime operation as much as possible. **[Delhi (MBA) 1998, 95]**

   Formulate and solve the given problem as a goal programming problem.

4. *XYZ* company produces two products : record player and taperecorders. The production of both products is done in two separate machine centres within the plant. Each record player requires two hours in machine centre 1 and one hour in machine centre 2. Each tape-recorder, on the other hand, requires one hour in mahcine centre 1 and three hours in mahcine centre 2. In addition, each product requires some in-process inventory. The per-unit inprocess inventory required is Rs. 50 for the record player and Rs. 30 for the tape-recorder. The firm has normal monthly operation hours of 120 for machine centre 1 and 150 for machine centre 2. The cost accounting department estimates that the average costs of per-hour operation are Rs. 80 and Rs. 20 for machine centres 1 and 2 respectively. The estimated profit per unit is Rs. 100 for the record player and Rs. 75 for the tape-recorder. According to the marketing department, the forecasted sales for the record player and the tape-recorder are 50 and 80 respectively for the coming month.
   The president of the firm has established the following goals for production in the next month, in ordinal rank of importance :

   (i) Limit the amount tied up in in-proces inventory for the month to Rs. 4,600,
   (ii) Achieve the sale goal of 80 tape-recorders for the month.
   (iii) Limit the overtime operation of machine centre 1 to 20 hours,
   (iv) Achieve the sales goal of 50 record players for the month.
   (v) Limit the sum of overtime operation for both machine centres,
   (vi) Avoid any under-utilization of regular operation hours of both machine centres.

   Formulate and solve the given problem as a goal programming problem.

5. *ABC computer company* produces three different types of computers : *Epic, Galaxie,* and *Utopia.* The production of all computers is conducted in a complex and modern assembly line. The production of an *Epic* requires 5 hours in the assembly line, a *Galaxie* requires 8 hours and a *Utopia* requires 12 hours. The normal operation hours of assembly line are 170 per month. The marketing and accounting departments have estimated that profits per unit for the three types of computers are Rs. 1,00,000 for the *Epic,* Rs. 1,44,000 for the *Galaxie,* and Rs. 2,52,000 for the *Utopia.* The marketing department further reports that the demand is such that the firm can expect to sell all the computers it produces in the month. The Chairman of the company has established the following goals, listed according to their importance :

(i) Avoid under-utilization of capacity in terms of regular hours of operation of the assembly line.

(ii) Meet the demand of the north-eastern sales district for five *Epics*, five *Galaxies*, and eight *Utopias* (differential weights should be assigned according to the net profit ratios among the three types of computers).

(iii) Limit overtime operation of the assembly line to 20 hours.

(iv) Meet the sales goal for each type of computers : *Epic*, 10 ; *Galaxie*, 12 ; and *Utopia*, 10 (again assign weights according to the relative profit function for each computer).

(v) Minimize the total overtime operation of the assembly line.

Formulate a goal programming model for the above problem and solve the problem through two iterations by simplex method of goal programming.

6. A hospital administration is reviewing departmental requests prior to the design of a new emergency room. At issue is the number of beds for each department. The current plans call for a 15,000 square feet facility. The hospital board has established the following goals in order of importance.

Department	No. of beds requested	Cost per bed (incl. equipment) (Rs.)	Area per bed (sq. ft.)	Peak requirement (Max. no. of partients at one time)
A	5	12,600	474	3
B	20	5,400	542	18
C	20	8,600	438	15

(i) Avoid overspending the Rs. 3,00,000 budget, (ii) Avoid plan requiring more than 15,000 sq. ft,

(iii) Meet plant requirement, (iv) Meet the departmental requests.

Formulate and solve (upto three iterations) the given problem as a goal programming problem.

7. The manager of the only record shop in a town has a decision problem that involves multiple goals. The record shop employs five full-time and four part-time salesmen. The normal working hours per month for a full-time salesman are 160, and 80 hours per month for a part-time salesman. According to performance records of the salesmen, the average sales has been five records per hour for full-time salesmen and two records per hour for part-time salesmen. The average hourly wage rates are Rs. 3 for full-time salesmen and Rs. 2. for part-time salesmen. Average profit from the sales of a record Rs. 1.50. In view of past records of sales, the manager feels that the sales goal for the next month should be 5,500 records. Since the shop is open six days a week, overtime is often required of salemen (not necessarily overtime but extra hours for the part-time salesmen). The manager believes that a good employer-employee relationship is an essential factor of business success. Therefore, he feels that a stable employment level with occasional overtime requirement is a better practice than an unstable employment level with no overtime. However, he feels that overtime of more than 100 hours among the full-time salesman should be avoided because of the declining sale effectiveness caused by fatigue. The manager has set the following goals :

(i) The first goal is to achieve a sales goal of 5,500 records for the next month.

(ii) The second goal is to limit the overtime of full-time salesmen to 100 hours.

(iii) The third goal is to provide job security to salemen. The manager feels that full utilization of employees' regular working hours (no layoffs) is an important factor for a good employer-employee relationship. However, he is twice as concerned with the full utilization of full-time salesmen as with the full utilization of part-time salesmen.

(iv) The last goal is to minimize the sum of overtime for both full-time and part-time salesmen. The manager desires to assign differential weights to the minimization of overtime according to the net marginal profit ratio between the full-time and part-time salesmen.

Formulate and solve the given problem as a goal programming problem.

8. *XYZ* Company plans to schedule its annual advertising campaign. The total advertising budget is set at Rs. 10,00,000. The firm can purchase local radio spots at Rs. 2,000 per spot, local television spots at Rs. 12.000 per spot or magazine advertising at Rs. 4,000 per insertion. The payoff from each advertising medium is a function of its audience size and audience characteristics. Let this payoff be defined as auidence points. Audience points for the three advertising vehicles are :

Radio	Television	Magazine
50 points per spot	250 points per spot	200 points per insertion.

The Advertising Manager of the firm has estabilished the following goals for the advertising campaign, listed in the order of importance :

(i The total budget should not exceed Rs. 10,00,000.

(ii) The contract with the radio and television station requires that the firm spend at least Rs. 3,00,000 for television and radio ads.

(iii) The company does not wish to spend more than Rs. 2,00,000 for magazine ads.

(iv) Audience points from the advertising campaign should be maximized.

(a) Formulate a goal programming model for this problem.

(b) Work through two iterations by simpelx method of goal programming. **[Delhi (MBA) 1999, 85]**

9. Mr. X has inherited Rs. 10,00,000 and seeks your advice concerning investing his money. You have determined that 10 percent can be earned on a bank account and 14 percent by investing in certificates of deposit. In real estate, you estimate an annual return of 15 percent can be earned, while on the stock market you estimate that 20 percent can be earned annually.

Mr. X has established the following goals in order of importance :

(i) Minimize risk, therefore he wants to invest not more than 35 percent in any one type of investment.

(ii) Must have Rs. 1,00,000 in the bank account to meet any emergency.

(iii) Maximum annual cash return.

Formulate and solve (upto three iterations only) a goal programming model that determines the amount of money which X should invest in each investment option.

10. Delta Hospital is a medium size many health care facilities (HCF) hospital located in a small city of Bihar. HCF is specialised in performing four types of surgery : T, A, H and C. The performance of these surgeries is constrained by three resources : operating room hours, recovery room bed hours and surgical service bed days. The director of HCF would like to achieve the following objectives in order of their importance, given the information below.

Resources	Type of Surgical Patients				Capacity
	P_1	P_2	P_3	P_4	
Operating Room	3	4	8	6	1,100 hrs.
Recovery Room	8	2	4	2	1,400 bed hrs.
Surgical service	4	6	2	4	40 bed days
Average contribution to profer (Rs.)	2,100	2,600	2,800	3,000	

P_1 : Achievment of at least Rs. 5,00,000 in profit in a specified period of time, given the available resources.

P_2 : Minimization of idle capacity of available resources.

Formulate and solve this problem as a GP problem.

11. A shoe manufacturer produces hiking boots and ski boots. Its manufacturing process consists of sewing and stitching. It has available 60 hours per week for the sewing process and 80 hours per week for the stitching process at normal capacity. The firm realizes profits of Rs. 150 per pair on hiking boots and Rs. 100 per pair on ski boots. It requires 2 hours of sewing and 5 hours of stitching to produce one pair of hiking boots and 3 hours of sewing and 2 hours of stitching to produce one pair of ski boots. The president of the company wishes to achieve the following goals, listed in order of their importance.

(i) Achieve the profit goal of Rs. 5,250 per week.

(ii) Limit of overtime operation of sewing centre to 30 hours.

(iii) Meet the sales goal for each type of boot — 25 liking boots and 20 ski boots.

(iv) Avoid, any underutilization of regular operation hours of the sewing centre Formulate and solve this problem as a goal programming model. **[Delhi (MBA) 2000]**

OBJECTIVE QUESTIONS

1. The use of GP model is preferred when
(a) goals are satisfied in an ordinal sequence. (b) goals are multiple incommensurable.
(c) more than one objective is set to achieve. (d) all of the above.

2. Deviational variables in GP model must satisfy the following conditions:
(a) $d_i^+ \times d_i^+ = 0$. (b) $d_i^+ - d_i^- = 0$. (c) $d_i^+ + d_i^- = 0$. (d) none of the above.

3. In GP at optimality, which of the following conditions indicate that a goal has been exactly satisfied:
(a) positive deviational variable is in the solution mix with a negative value.
(b) both positive and negative deviational variables are in the solution mix.
(c) both positive and negative deviational variables are not in the solution mix.
(d) none of the above.

4. In GP problem, goals are assigned priorities such that
(a) higher priority goals must be achieved before lower priority goals.
(b) goals may not have equal priority.
(c) goals of greatest importance are given lowest priority.
(d) all of the above.

5. In GP problem, a constraint having unachieved variable is expressed as:
(a) an equality constraint. (b) a less than or equal to type constraint.
(c) a greater than or equal to type constraint. (d) all of the above.

6. For applying a GP approach, decision-maker must
(a) set targets for each of the goals.
(b) assigan pre-emptive priority to each goal.
(c) assume that linearity exists in the use of resources to achieve goals.
(d) all of the above.

7. Consider a goal with constraint: $g_1(x_1, x_2, \ldots, x_n) + d_1^- - d_1^+ = b_1$ and the term $2d_1^- + 3d_1^+$ in the objective function, the decision-maker
(a) prefers $g_1(x_1, x_2, \ldots, x_n) \geq b_1$, rather than $\leq b_1$.
(b) prefers $g_1(x_1, x_2, \ldots, x_n) \leq b_1$ rather than $\geq b_1$.
(c) not concerned with either $\leq$ or $\geq$.
(d) none of the above.

8. Consider a goal with constraint : $g_1(x_1, x_2, \ldots, x_n) + d_1^- \geq b_1$, $d_1^- \geq 0$ with d_1^- in the objective function. Then
(a) the goal is to minimize under-achievement.
(b) the constraint is active provided $d_1^- \geq 0$.
(c) both (a) and (b).
(d) none of the above.

9. Goal programming
(a) requires only that decision-maker knows whether the goal is direct profit maximization or cost minimization.
(b) allows you to have multiple goals, with or without priorities.
(c) is an approach to achieve goal of a solution to all IP problems.
(d) none of the above.

10. In simplex method of goal programming, the variable to enter the solution mix is selected with
(a) lowest priority row and most positive $z_j - c_j$ value in it.
(b) lowest priority row and largest negative $z_j - c_j$ value in it.
(c) highest priority row and most positive $z_j - c_j$ value in it.
(d) highest priority row and most negative $z_j - c_j$ value in it.

Answers

1. (d) **2.** (a) **3.** (c) **4.** (a) **5.** (c) **6.** (d) **7.** (a) **8.** (c) **9.** (a) **10.** (c).

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UNIT 3

PROBABILITY AND STATISTICAL ANALYSIS FOR MANAGEMENT DECISIONS

CONTAINING :

14

REVIEW OF PROBABILITY THEORY

14.1. INTRODUCTION

So far we were concerned with the problems in which distances, times and other variables were almost always given precise numerical values. In many situations, however, there are quantities which are subject to random variation, like monthly level of sales in a supermarket shop. Since it is not always necessary to construct a ***deterministic*** model, we can choose to adopt a ***stochastic*** model. In order to apply the optimization principles of Operations Research to systems whose details we do not fully understand or cannot easily analyse, we borrow ideas from statistics and build-up stochastic models.

In this chapter, we develop the concept of probability because this is one of the main statistical tools used in operations research.

The student who feels confident of his abilities in the area of probability theory may skin, or skip altogether, the entire chapter. But, the student who has no previous knowledge to the subject is well advised to go through this chapter carefully.

14.2. CONCEPT OF UNCERTAINTY

Consider the following experiment.

A person X tosses a coin and Y also tosses a similar coin simultaneously and observe the result.

It is *uncertain* to say :

(i) Whether both heads come up or *(ii)* both tails come up or *(iii)* one head and one tail come up.

The situation of uncertainty can also be observed by considering two sets A and B in three different situations as shown in the followng diagram :

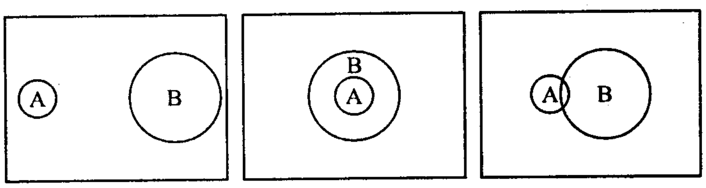

Fig. 14.1.

In situations (*i*) and (*ii*), it is certain that $x \in A \Rightarrow x \notin B$; $x \in A \Rightarrow x \in B$, respectively; while in situation (*iii*) if $x \in A$ we cannot say whether the element $x \in B$ or $x \notin B$ which is the situation of uncertainty.

Before giving a formal definition of probability, it is necessary to introduce the concepts of *probability, sample space and events.*

14.3. CONCEPT OF PROBABILITY

In our daily life, we often come across sentences like :

(*i*) It is impossible that he may refuse to do my work. (*ii*) Probability, it will rain tomorrow.

Here, in sentence (*i*), we observe that the probability of doing work is unity and, therefore, probability of refusing to do work will be zero. Consequently, in sentence (*ii*) the probability of raining lies between *zero* and *unity.*

14.4. SAMPLE SPACE

A sample space can be defined as the set of all possible outcomes of an experiment and is denoted by S.

Suppose all possible outcomes of an experiment are denoted by $e_1, e_2, e_3, \ldots e_n$, which are such that no two or more of them can occur simultaneously, and exactly one of the outcomes $e_1, e_2, e_3, \ldots, e_n$ must occur whenever the experiment is performed.

Definition. *The set* $S = \{e_1, e_2, e_3, \ldots, e_n\}$ *is called a* ***sample space*** *of an experiment satisfying the following two conditions :*

(*i*) *Each element of the set S denotes one of the possible outcomes.*

(*ii*) *The outcome is one and only one element of the set S whenever the experiment is performed, e.g., in tossing a coin, sample space consists of head or tail,* i.e. $S = \{H, T\}$;
when two coins are tossed simultaneously, $S = \{HH, HT, TH, TT\}$;
when three coins are tossed simultaneously, $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$.

For example, we perform an experiment of rolling a six-faced die whose faces are numbered as e_1, e_2, e_3, e_4, e_5 and e_6. Whenever the die comes to rest, one of the numbers e_1, e_2, e_3, e_4, e_5, or e_6 appears on the top face. If we ask the question, "Which number came out on the top" ? there are six possible answers. Each of these possible answers is a possible outcome of the experiment. The set $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ whose every member designates one of these possible outcomes, is called a ***Sample Space*** for the experiment.

Instead of asking, "Which number come out on the top" ? we might ask, "Is the number that comes out on top odd or even" ? Then the set of words {odd, even}, designating two possible outcomes, is also a sample space.

Example 1. *From an urn containing 4 balls of different colours, i.e., Red (R), Blue (B), Yellow (Y) and Green (G), two balls are drawn; (i) simultaneously (ii) one after the other with replacement. Define the sample space in both the cases.*

Solution. (*i*) The number of ways of drawing 2 balls out of 4 balls = ${}^4C_2 = 6$.

The sample space is the set of all possible combinations of two balls of different colour.

Hence, the sample space in first case is : $S = \{BR, YR, GR, BY, BG, GY\}$. **Ans.**

(*ii*) In this case, the balls are drawn one after the other with replacement. Therefore, the combination of the same colour balls is also possible.

Hence, the sample space is given by the set

$$S = \{GG, GY, GB, GR, YR, YB, YY, YG, BB, BR, BY, BG, RR, RB, RY, RG\}.$$ **Ans.**

14.5. ELEMENTARY EVENTS

Definition. *The element or the point of a sample space associated with an experiment are called the elementary events of the experiment.*

It is the obvious fact that no two elementary events can occur simultaneously, and exactly one of these must occur in a single trial. For example, a marksman cannot hit and miss the target simultaneously, and further a single shot must either hit the target or miss it.

14.6. ACCEPTABLE ASSIGNMENT OF PROBABILITIES TO ELEMENTARY EVENTS

Consider a sample space S of an experiment consisting of n elementary events $e_1, e_2, e_3, \ldots, e_n$.

Definition. *If to each elementary event* $e_i \in S$, $i = 1, 2, \ldots, n$, *we assign a real number* $P(e_i)$ *called the* ***probability*** *of an elementary event* e_i *such that :*

(*i*) *the probability of each* $e_i (i = 1, 2, \ldots, n)$ *is a non-negative real number, i.e.,* $P(e_i) \geq 0$ for $i = 1, 2, \ldots, n$;

(*ii*) *the sum of the probabilities assigned to all elementary events of the sample space S is unity, i.e.,*

$$P(e_1) + P(e_2) + \ldots + P(e_n) = 1;$$

then such an assignment of probabilities to elementary events is called the acceptable assignment of probabilities to elementary events.

For example, in a coin tossing experiment, the sample space is $S = \{H, T\}$ where H and T are used to stand for *Head* and *Tail*, respectively.

It is evident that

(*i*) If $P(H) = p$ (say) and $P(T) = 1 - p$ where $0 \le p \le 1$, then it is an acceptable assignment of probabilities.

(*ii*) $P(H) = -\frac{1}{2}$ and $P(T) = \frac{3}{2}$ is not an acceptable assignment because probability cannot be less than zero and greater than unity although the sum $P(H) + P(T) = 1$.

(*iii*) $P(H) = 0.50$ and $P(T) = 0.49$ is also not an acceptable assignment because $P(H) + P(T) \neq 1$.

14.7. NATURAL ASSIGNMENT OF PROBABILITIES TO ELEMENTARY EVENTS

If each elementary event is assigned the same probability, *i.e.*,

$$P(e_1) = P(e_2) = P(e_3) = \ldots = P(e_n) = 1/n \text{ , [because } P(e_1) + P(e_2) + \ldots + P(e_n) = 1]$$

then such an assignment is called ***natural assignment.***

For example, in a coin tossing experiment $S = \{H, T\}$, $P(H) = P(T) = \frac{1}{2}$ satisfying $P(H) + P(T) = 1$ is the natural assignment.

14.8. EVENTS

Consider, for example, the experiment of throwing a six-faced die. The sample space is $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, where the appearance of face numbered 1 is denoted by e_1, face numbered 2 is denoted by e_2 etc. If we are interested in the event of appearing even numbered faces up, then the event will have the outcomes e_2, e_4, e_6 only and hence this event is denoted by $E_1 = \{e_2, e_4, e_6\}$, which is the subset of S. We now watch for the event, "The face number that turns up is divisible by 3". This event will be denoted by $E_2 = \{e_3, e_6\}$ which is also a subset of S. Thus we may define the event as follows :

Definition 1. ***(Event)*** *Every subset of a sample space S of an experiment is called an* ***event*** *generally denoted by E.*

In particular, the sample space S itself is called the ***certain event*** which is obviously the subset of itself. And the impossible event will be denoted by the empty subset ϕ of sample space S.

Simple Event. Furthermore, among the subsets of S some subsets are containing only one member. Here is a complete list of these one-member subsets $\{e_1\}$, $\{e_2\}$, $\{e_3\}$, $\{e_4\}$, $\{e_5\}$, and $\{e_6\}$. Because the one-member events play an important role in the theory of probability, so we give them a special name.

Definition 2. ***(Simple Event).*** *Any event that contains only one member of a sample space is called a* ***simple event*** *in this space.*

For example, let a die be rolled once and A be the event that face 3 is turned up, then A is called a simple event.

Definition 3. ***(Compound Events).*** *When an event is decomposable into a number of simple events, then it is called a* ***compound event.***

For example, the event, "the sum of the two numbers shown by the upper faces of the two dice is seven in the simultaneous throw of the two unbiased dice", is a compound event as it can be decomposed into the following simple events (1, 6), (6, 1), (5, 2), (2, 5), (4, 3), (3, 4).

Definition 4. ***(Equally Likely Events).*** *Events are said to be equally likely if there is no reason to expect any one in preference to any other i.e., when the probability of happening of two or more events is the same, they are called* ***equally likely events.***

For example, if a coin is tossed then there may be either head up or tail up *i.e.*, there is no reason to expect the occurrence of tail in preference to head up.

Definition 5. ***(Mutually Exclusive Events).*** *Events are said to be* ***mutually exclusive*** *when the happening of one excludes the happening of the other i.e., no any two events can occur simultaneously.*

For example, when a coin is tossed, event of throwing head and event of throwing tail are mutually exclusive events.

Event of 'rolling a face numbered less than 3' and event of 'rolling even numbered face on a die' are not mutually exclusive.

Definition 6. ***(Dependent and Independent Events).*** *Two events are said to be* ***independent*** *when occurrence of one has no effect on the probability of other.*

Events are said to be ***dependent****, if they are not independent.*

Illustrative Examples

Example 2. *If a bag contains seven balls, and one ball is drawn from it and is replaced back, then a second ball is drawn from it; the probability of the second drawing is independent of the first and so the two drawings (events) will be independent. Again if first ball is not replaced back, then a second ball is drawn from it; the prob. of the second drawing is dependent of the first and so the two (events) are dependent.*

Example 3. *If a die is rolled twice, event of getting face number 5 in first tossing and event of getting face number 3 in the second tossing are independent.*

14.9. SET NOTATIONS FOR EVERYDAY LANGUAGE

$E \subset S$	:	*Event E.*
$\underline{e} \in S$	:	*e* is an outcome of an experiment whose sample space is *S*.
$\bar{E}$	:	Complementary event of event *E*.
$e \in \underline{E}$	:	Event *E* occurs.
$e \in \bar{E}$	:	Event *E* does not occur.
$e \in E \cup F$	:	Event *E* or even *F* occurs.
$e \in E \cap F$	:	Event *E* and event *F* occurs.
$E = \phi$	:	Event *E* is impossible.
$E = S$	:	Event *E* is a certain event.
$F \subset E$	:	If event *E* occurs, then event *F* will also occur.
$E \cap F = \phi$	:	If event *E* occurs, then event *F* does not occur, *i.e.*, *E* and *F* are independent events.

14.10. PROBABILITY OF AN EVENT

Definition. ***Probability.*** *With each event E_i in a finite sample space S, we associate a real number, say $P(E_i)$, called the probability of an event E_i satisfying the following conditions :*

(*i*) $0 \le P(E_i) \le 1$. *This implies that the probability of an event is always non-negative and can never exceed unity.*

(*ii*) $P(E_1 \cup E_2 \cup E_3 \ldots \cup E_k) = P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_k)$ *where* $E_1, E_2, E_3, \ldots, E_k$ *are mutually exclusive events in S.*

(*iii*) $P(S) = 1$, *where the event S is the entire sample space called the* ***certain event.***

Q. What are the axioms of probability ? **[Kanpur 96]**

14.11. FREQUENCY INTERPRETATION (Classical Definition of Probability)

If n denotes the number of times the experiment is performed and m the number of successful occurrences of the event E_i in the n trials, then $P(E_i)$ can be defined as

$$P(E_i) = \lim_{n \to \infty} \frac{m}{n} \quad \text{(finite and unique)},$$

provided the outcomes of event E_i are :

(*i*) **exhaustive :** which implies (as in a coin tossing experiment) that it must fall either head or tail but cannot stand on edge;

(*ii*) **mutually exclusive :** which implies that head and tail must not occur simultaneously. We also know the fact that a marksman cannot hit and miss the target simultaneously, and moreover a single shot must either hit the target or miss it; and

(*iii*) **equally likely :** which means that head and tail both have equal chances to occur.

Also, the ratio m/n does fulfil the following four conditions of probability :

(*a*) $0 \le m/n \le 1$.

(*b*) $\dfrac{m_1 + m_2 + m_3 + \ldots + m_k}{n} = \dfrac{m_1}{n} + \dfrac{m_2}{n} + \dfrac{m_3}{n} + \ldots + \dfrac{m_k}{n}$, where $E_1, E_2, E_3, ..., E_k$ are independent (disjoint) events and each occurs $m_1, m_2, m_3, ..., m_k$ number of times, respectively, in n number of trials.

(*c*) $n/n = 1$, since the event E_i must occur every time the experiment is performed, therefore $m = n$.

(*d*) $0/n = 0$, $m = 0$, since the event E_i cannot occur in any trial.

Actual probability of an event E_i cannot be determined by classical definition because of $n \to \infty$. Although it is possible to estimate the probability of an event experimentally by drawing a graph between n and $m/n \to \infty$, the ratio m/n becomes very nearly constant which will be the estimated probability of an event.

14.11-1. The Playing Card Example

Example 4. *A playing card is chosen at random from a pack of 52 cards. The 52 possible outcomes to this trial can be displayed by a diagram called the* ***sample space (or outcome space)*** *diagram.*

Sample Space Diagram

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Hearts	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥
Diamonds	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦
Clubs	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣
Spades	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠

Now the following questions may be answered.

Question 1. *What is the probability of choosing the 3 of Diamonds ?*

Answer. In view of the argument that for every 52 trials, one of the cards will be the 3 of *Diamonds*, we deduce that the *probability of the 3 of Diamonds*, which is written as ***P*(3 of diamonds)**, is 1/52. It is worthnoting that while repeating a trial, the conditions which existed originally must be restored. In this case, the card chosen is considered to be returned back to the pack each time otherwise the trial is not an exact repetition of the previous one.

Question 2. *What is the probability of choosing any* 3 ?

Answer. We know that an average 4 out of every 52 cards chosen will be a '3' as there are four '3'*s* in the whole pack. So the answer will be 4/52.

We can also deduce this answer from the above as the fraction of trials resulting in 'any 3' is the sum of the fractions of times the 3 of *Hearts*, the 3 of *Diamonds*, the 3 of *Clubs* and the 3 of *Spades* occur. Hence we have

$$P(\textbf{any 3}) = \frac{1}{52} + \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{4}{52}, \text{ as before.} \qquad (...14.1)$$

Question 3. *What is the probability of not getting a* 3 ?

Answer. Since there are 48 cards which are not '3'*s*, therefore, *P* (not 3) will be 48/52.

Alternatively, it can be argued that on any trial the card chosen is either a '3' or not a '3'. So the fraction of trials on which a '3' occurs and the fraction on which a '3' does not occur must add up to 1.

$$P(\textbf{not 3}) = 1 - P(\text{any } 3) = 1 - \frac{4}{52} = \frac{48}{52}, \text{ as before.} \qquad ...(14.2)$$

In general, if the letter A stands for any event, then

$$P(\textbf{not } A) = 1 - P(A) \qquad ...(14.3)$$

Question 4. *What is the probability of choosing any 3 or any Diamond or both ?*

Solution. It can be seen from the following diagram that there are 16 playing cards which are either a '3' or a Diamond, or Both. Therefore, *P* (3 or Diamond or Both) = 16/52.

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Hears	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥
Diamonds	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦
Clubs	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣
Spades	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠

Alternatively, the answer can also be obtained by as well as simply counting the dots in the above diagram. The set 'any 3 or any Diamond or Both' is the union of the sets 'any 3 which contains 4 cards, and 'any Diamond', which contains 13 cards. The number of cards in their union is equal to the sum of these numbers minus the number of cards in the space where they overlap. Any points in this space, called the *intersection* of the two sets, is counted here twice, once in each set. Dividing all these numbers (4, 13 and 1) by 52 to change them into probabilities, we observe that—

P(any 3 or any Diamond or Both) = P(any 3) + P(any Diamond) – P(3 of Diamond)

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}, \text{ as we obtained earlier.} \quad ...(14.4)$$

In general, if the letters A and B stands for any two events, then above equation can be written as

$$P(A \text{ or } B \text{ or Both}) = P(A) + P(B) - P(A \cap B). \quad ...(14.5)$$

Question 5. *What is the probability of choosing any* 3 *or say* 7 ?

Answer. The number of cards which are any '3 or any 7' is 8 and therefore the desired answer is 8/52.

Alternatively, using the above formula with A corresponding to the event 'any 3' and B to the event 'any 7' we get.

P(any 3 or any 7 or Both) = P (any 3) + P (any 7) – P (any 3 and any 7)

$$= \frac{4}{52} + \frac{4}{52} - \frac{0}{52} = \frac{8}{52}. \quad ...(14.6)$$

Remember that P(any 3 and any 7) is zero because a selected card cannot be both a '3' and a '7'. So the events which cannot occur simultaneously like this are called mutually exclusive and the formula (14.5) will then reduce to a simple addition rule for their probabilities. An example of this can be seen in the answer of question 2.

Remember. 1. For any event A, $0 \le P(A) \le 1$, *i.e.*, the scale of probability extends from 0 to 1.

2. If A is an ***impossible*** event, then $P(A) = 0$.

3. If A is a ***certain*** event, then $P(A) = 1$.

Odds in favour or against an event. Instead of saying that the probability of the happening of an event is $\frac{m}{m+n}$, it is sometimes stated that odds are $m : n$ in favour of the event or $n : m$ against the event.

14.11-2. Illustrative Examples

Example 5. *What is the chance of throwing a* 3 *with an ordinary die* ?

Solution. Since the die may fall in 6 ways and only one is favourable, therefore required probability = 1/6.

Example 6. *In a class of 32 students, 17 are girls and the rest are the boys. Find the probability that a student selected will be a boy.*

Solution. No. of boys = 32 – 17 = 15.

$\therefore$ The prob. that a boy is selected $= \dfrac{\textit{no. of fav. cases}}{\textit{total no. of cases}} = \dfrac{15}{32}$. **Ans.**

Example 7. *From a bag containing 10 white and 15 red balls, a ball is drawn at random. What is the probability that it is a white ball ?*

Solution. Total no. of balls in the bag = 10 + 15 = 25. No. of white balls = 10.

$\therefore$ Prob. of getting a white ball $= \dfrac{\textit{no. of fav. cases}}{\textit{total no. of cases}} = \dfrac{10}{25} = \dfrac{2}{5}$. **Ans.**

Example 8. *A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning this bet ?*

Solution. One card can be drawn from a pack of 52 cards in ${}^{52}C_1 = 52$ ways.

No. of ways in which a card can be a *spade* or an *ace* = 13 + 3 = 16.

$\therefore$ The prob. of winning the bet $= \dfrac{16}{52} = \dfrac{4}{13} = \dfrac{4}{4+9}$.

$\therefore$ Odds against his winning this bet = 9 : 4. **Ans.**

Example 9. *From a bag containing 10 black and 5 white balls, a ball is drawn at random, what is the probability that it is white ?*

Solution. Since the total number of balls in the bag are $10 + 5 = 15$, the number of ways of drawing one ball from the bag $= {}^{15}C_1 = 15$.

Also, since the total number of white balls is 5, the number of ways of drawing one white ball from the bag $= {}^{5}C_1 = 5$.

Hence the probability of drawing one white ball from the bag $= 5/15 = 1/3$. **Ans.**

Example 10. *Find the probability of obtaining a total of 6 in a single throw of two dice.*

Solution. The number of ways in which total of numbers with two dice can come is $6 \times 6 = 36$. Also, the total sum 6 can be obtained on two dice in following 5 ways : $1 + 5, 2 + 4, 3 + 3, 4 + 2, 5 + 1$.Hence the probability of obtaining a total of 6 in a single throw of two dice $= 5/36$. **Ans.**

EXAMINATION PROBLEMS

1. What is the sample space for the experiment which consists of drawing 2 balls with replacement from an urn containing 8 balls ? The balls are numbered 1 through 8.
   [**Ans.** $S = \{(x_1, x_2) : x_i = 1, 2, ..., 8;\ i = 1, 2\}$.]
2. In above problem, define the events as subsets : A : The first ball is green, B : The second balls is green, C : Both the balls are green.
   [**Ans.** $A = \{(x_1, x_2) : x_1 = 1, 2, 3;\ x_2 = 1, 2, ..., 8\}$, $B = \{(x_1, x_2) : x_1 = 1, 2, ..., 8;\ x_2 = 1, 2, 3\}$,
   $C = \{(x_1, x_2) : x_i = 1, 2, 3;\ i = 1, 2\}$.]
3. What is the sample space for the experiment which consists of drawing one ball from an urn containing 8 balls of which 3 are green and 5 are red ? The balls have been numbered 1 through 8.
   [**Ans.** $S = \{1, 2, ..., 8\}$.]
4. In problem 3 above, define the events as subsets : A : a green ball is drawn, B : a red ball is drawn.
   [**Ans.** $A = \{1, 2, 3\}$; $B = \{4, 5, 6, 7, 8\}$.]
5. What is the probability of getting 2 tails and 2 heads when 4 coins are tossed ?
   [**Ans.** 3/8]
6. What is the probability of obtaining 9, 10 and 11 points with 3 dice ?
   [**Ans.** $\frac{25}{216}, \frac{27}{216}, \frac{27}{216}$.]

14.12. THE ADDITION LAW OF PROBABILITY

If the events $E_1, E_2, E_3, ..., E_n$ are mutually exclusive in pairs, then

$$P(E_1 \cup E_2 \cup E_3 \ldots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \ldots + P(E_n)$$

Theorem. *If $p_1, p_2, ..., p_n$ be separate probabilities of n mutually exclusive events, then the probability of the happening of any one of these events is given by $p = p_1 + p_2 + p_3 + \ldots + p_n$.*

Proof. Let $A_1, A_2, \ldots, A_n$ be n mutually exclusive events.

Let N be the number of cases which are equally likely, mutually exclusive and exhaustive.

Out of these N cases, let

No. of cases favourable to event $A_1 = m_1$

No. of cases favourable to event $A_2 = m_2$

... ...

No. of cases favourable to event $A_n = m_n$.

But $A_1, A_2, ..., A_n$ are mutually exclusive. Therefore, $m_1, m_2, ..., m_n$ are distinct and non over-lapping.

$\therefore$ Total number of cases which are favourable to either A_1, or A_2 ... or A_n are $= m_1 + m_2 + \ldots + m_n$

$\therefore$ The probability 'p' of any one of the events happening $= \dfrac{m_1 + m_2 + \ldots + m_n}{N}$

$$= \frac{m_1}{N} + \frac{m_2}{N} + \ldots + \frac{m_n}{N} = P(A_1) + P(A_2) + \ldots + P(A_n) \text{ [by def.]}$$

$\therefore$ $p = p_1 + p_2 + \ldots + p_n$ which proves the theorem.

Remember. Probability of occurrence of at least one of the two non-mutually exclusive events is given by
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

14.13. THE CONDITIONAL PROBABILITY

Consider the two events E_1 and E_2 in a sample space S. Here E_1 represents the event that has occurred m_1 number of times in n (large) number of trials, and E_2 represents the event that has occurred m_2 number of times out of these m_1 number of occurrences of E_1. Therefore, by using the classical definition of probability, we can find the probability of the combined happening of E_1 and E_2 in the same trial as
$$P(E_1 \cap E_2) = \frac{m_2}{n} = \frac{m_1}{n} \cdot \frac{m_2}{m_1}. \qquad \text{...(14.7)}$$
Obviously, $m_1/n = P(E_1)$, but the relative frequency m_2/m_1 can approximately be taken as the *conditional probability of occurrence* of event E_2 given that E_1 has occurred $[P(E_1) \neq 0]$, which is denoted by $P(E_2 \mid E_1)$.

Now, (14.7) becomes
$$P(E_1 \cap E_2) = P(E_1)\, P(E_2 \mid E_1) \qquad \text{...(14.8)}$$
That is, *the probability that both* E_1 and E_2 *occur is equal to the prob. that* E_1 *occurs* m_1 *times the prob. that* E_2 *occurs given that* E_1 *has occurred.*

The above discussion suggests that we define the *conditional* probability of E_2 given E_1,
$$P(E_2 \mid E_1) = P(E_1 \cap E_2)/P(E_1), \text{ where } P(E_1) > 0. \qquad \text{...(14.9)}$$
Similarly, we can get
$$P(E_1 \mid E_2) = P(E_1 \cap E_2)/P(E_2), \text{ where } P(E_2) > 0. \qquad \text{...(14.10)}$$
Furthermore, $P(E_2 \mid E_1)$, the conditional probability of E_2 satisfies the following four properties.

P_1. $0 \le P(E_2 \mid E_1) \le 1$.

P_2. If E_2 is an event which cannot occur, then $P(E_2 \mid E_1) = 0$.

P_3. If the event E_2 is the entire sample space S, then $P(S \mid E_1) = 1$.

P_4. If E_2 and E_3 are two independent events in S, then $P(E_2 \cap E_3 \mid E_1) = P(E_2 \mid E_1) + P(E_3 \mid E_1)$.

In case the occurrence of E_1 does not affect the occurrence of E_2, we have
$$P(E_2 \mid E_1) = P(E_2). \qquad \text{...(14.11)}$$
Thus, E_1 and E_2 are independent if and only if,
$$P(E_1 \cap E_2) = P(E_1)\, P(E_2). \quad \text{[using results of (14.8) and (14.11) here]}$$
We now state an important law of probability :

The law of total probability. For any n events $E_1, E_2, ..., E_n$.
$$P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n) = P(E_1) \,.\, P(E_2 \mid E_1) \,.\, P(E_3 \mid E_1E_2) \,.\, P(E_n \mid E_1E_2 \ldots E_{n-1}).$$
If the events $E_1, E_2, ..., E_n$ are independent, then
$$P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n) = P(E_1).P(E_2) \ldots P(E_n)$$

Q. 1. Define conditional probability, and outline its properties.

2. A and B are any two events and the probability $P(B) \neq 1$, prove that $P(A \mid B) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$. **[IAS (Main) 97]**

14.13-1. More Illustrative Examples

Example 11. *A box contains 25 parts, of which 10 are defective. Two parts are drawn at random from the box. What is the probability that both are good ?* **[Rohilkhand 92]**

Solution. Let E be the event *'the first part drawn is good'*, and F be the event *'the second part drawn is good'*. Obviously, $E \cap F$ is the event *'both are good'*. Therefore,
$$P(E \cap F) = P(E)\, P(F \mid E) = \frac{15}{25} \cdot \frac{14}{24} = \frac{7}{20}.$$
Ans.

Example 12. *A coin is tossed three times in succession. If E is the event that there are at least two heads, and F is the event that first throw gives a head, find* $P(E \mid F)$.

Solution. Sample space of this experiment is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
Obviously, the probability, of each sample point is 1/8.
The sample spaces S_1 and S_2 of events E and F, respectively, are :

$$S_1 = \{HHH, HHT, HTH, THH\}. \text{ and } S_2 = \{HHH, HHT, HTH, HTT\}.$$

Therefore, $P(E) = P(F) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$.

Since the events E and F happen simultaneously (*i.e.*, $E \cap F$) at the common points— HHH, HTH, HHT,

$$\therefore \qquad P(E \cap F) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$$

Hence $P(E \mid F) = P(E \cap F)/P(F) = \frac{3/8}{1/2} = \frac{3}{4}$. **Ans.**

Example 13. *From a pack of 52 cards, two are drawn in two successive trials. Find the probability of both being aces if (i) card is not replaced after the draw, (ii) card is replaced after the first draw.*

Solution. (*i*) $P(E_2 \mid E_1) =$ Prob. of getting an ace in second draw $= 3/51$ ($\because$ Ist card was not replaced)

$$\therefore P(E_1 \cap E_2) = P(E_1)\, P(E_2 \mid E_1) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221}.$$

(*ii*) If the card is replaced back after first draw, $P(E_1 \cap E_2) = \frac{4}{52} \times \frac{4}{52} = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

Example 14. *In a simultaneous toss of two coins, find the prob. of*: (a) 2 *tails*, (b) *exactly* 1 *tail*, (c) *no tails.*

Solution. Sample space $S = \{HH, HT, TH, TT\}$.
(a) P(2 tails) = 1/4, (b) P(exactly 1 tail) = 2/4 = 1/2., (c) P(no tails) = 1/4.

Example 15. *A card is drawn at random from a well-shuffled deck of* 52 *cards. Find the probability of :*
(a) *a king,* (b) *a heart,* (c) *a jack, queen or a king,* (d) *a black card.*

Solution. A card can be drawn from a deck of 52 cards in $^{52}C_1 = 52$ ways.

(a) $P(\text{a king}) = \frac{^{4}C_1}{^{52}C_1} = \frac{4}{52} = \frac{1}{13}$, (b) $P(\text{a heart}) = \frac{^{13}C_1}{^{52}C_1} = \frac{13}{52} = \frac{1}{4}$,

(c) $P(\text{a jack, queen or king}) = \frac{^{12}C_1}{^{52}C_1} = \frac{12}{52} = \frac{3}{13}$, (d) $P(\text{a black card}) = \frac{^{26}C_1}{^{52}C_1} = \frac{26}{52} = \frac{1}{2}$.

Example 16. *A and B are two events such that* $P(A) = 0.42, P(B) = 0.48,$ *and* $P(A \text{ and } B) = 0.16$. *Determine* :
(a) $P(not\, A)$, (b) $P(not\; B)$, (c) $P(A\; or\; B)$.

Solution. (a) $P(\text{not } A) = 1 - P(A) = 1 - 0.42 = 0.58$, (b) $P(\text{not } B) = 1 - P(B) = 1 - 0.48 = 0.52$,
(c) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 0.42 + 0.48 - 0.16 = 0.74$.

Example 17. *Three coins are tossed once.*
(a) *Write the points of S,* (b) *Find the probability of:* (*i*) 3 *heads,* (*ii*) *exactly* 2 *heads,* (*iii*) *at least* 2 *heads.*

Solution. (a) $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT,\}$
(b) (*i*) P(3 heads) = 1/8, (*ii*) P(exactly 2 heads) = 3/8, (*iii*) P(at least 2 heads)

$$= P(\text{exactly 2 heads}) + P(\text{3 heads}) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}.$$

Example 18. *An urn contains* 9 *red,* 7 *white and* 4 *black balls. A ball is drawn at random. What is the probability that the ball drawn will be :* (a) *red* ? (b) *white* ? (c) *white or black* ? (d) *red or black* ?

Solution. Total no. of balls = 9 + 7 + 4 = 20.

A ball can be drawn from the urn containing 20 balls in $^{20}C_1$ ways.

(a) $\therefore P(\text{red ball}) = \frac{^{9}C_1}{^{20}C_1} = \frac{9}{20}$ (b) $P(\text{a white ball}) = \frac{^{7}C_1}{^{20}C_1} = \frac{7}{20}$

(c) $P(\text{white or black}) = \frac{^{11}C_1}{^{20}C_1} = \frac{11}{20}$ (d) $P(\text{red or black}) = \frac{^{13}C_1}{^{20}C_1} = \frac{13}{20}$.

Example 19. *From a bag containing 7 red and 5 black marbles, one is drawn at random. what is the probability of the marble being black.*

Solution. Total marbles in a bag = 7 + 5 = 12 and one marble can be drawn in ${}^{12}C_1$ ways.

Prob. (marble being black) $= \dfrac{{}^{5}C_1}{{}^{12}C_1} = \dfrac{5}{12}$.

Example 20. *A card is drawn from a well-shuffled deck of 52 cards. Find the probability of drawing :*

(a) *a black king,* (b) *a jack, queen, kind or ace,* (c) *a card which is neither a heart nor a king,* (d) *a spade or a club.*

Solution. A card can be drawn from a pack of 52 *cards in* ${}^{52}C_1$ ways.

(a) P (a black king) $= \dfrac{{}^{2}C_1}{{}^{52}C_1} = \dfrac{2}{52} = \dfrac{1}{26}$ (b) P (a jack, queen, king or ace) $= \dfrac{{}^{16}C_1}{{}^{52}C_1} = \dfrac{16}{52} = \dfrac{4}{13}$

(c) P (a card which is neither a heart nor a king) $= \dfrac{{}^{36}C_1}{{}^{52}C_1} = \dfrac{36}{52} = \dfrac{9}{13}$

(d) P (a spade or a club) $= \dfrac{{}^{26}C_1}{{}^{52}C_1} = \dfrac{26}{52} = \dfrac{1}{2}$.

Example 21. *In a single throw with two dice find the chances of throwing* : (*i*) *eight* (*ii*) *eleven.*

Solution. Two dice can be thrown in 6 × 6 = 36 ways.

(*i*) A total of eight can be obtained as (6, 2), (2, 6), (5, 3), (3, 5), (4, 4). Therefore, P (a total of 8) = 5/36.

(*ii*) A total of 11 can be obtained as (6, 5), (5, 6).

$\therefore$ P (a total of 11) = 2/36 = 1/18.

Example 22. *In a single throw of two dice, determine the prob. of getting :*

(a) *a total of* 2, (b) *a total of* 12, (c) *a total of* 7 *or* 9.

Solution. Two dice can be thrown is 6 × 6 = 36 ways.

(a) A total of 2 can be obtained as (1, 1) = 1 way. Therefore, P (a total of 2) = 1/36.

(b) A total of 12 can be obtained as (6, 6) = 1 way. Therefore, P (a total of 12) = 1/36.

(c) A total of 7 can be obtained as (6, 1), (1, 6), (5, 2), (2, 5),(4, 3), (3, 4).

A total of 9 can be obtained as (6, 3), (3, 6), (5, 4),(4, 5).

Therefore, A total of 7 or 9 can be obtained in 10 ways. Hence required prob. = 10/36 = 5/18.

Example 23. *Given two independent events A,B such that P(A) = 0.30 and P(B) = 0.60.*

Determine : (a) P (*A and B*), (b) P (*A and not B*), (c) P(*not A and B*,) (d) P (*neither A nor B*), (e) P (*A or B*).

Solution. (a) P (A and B) = P (A). P (B) = 0.30 × 0.60 = 0.18

(b) P (A and not B)= $P(A).P(\bar{B})$ = 0.30 × (1 – 0.60) = 0.30 × 0.40 = 0.12

(c) P (not A and B) = $P(\bar{A}).P(B)$ = (1 – 0.30) × 0.60 = 0.70 × 0.60 = 0.42

(d) P (neither A nor B) = $P(\bar{A}).P(\bar{B})$ = (1 – 0.30) (1 – 0.60) = 0.70 × 0.40 = 0.28.

(e) P (A or B) = 1 – P (neither A nor B)= 1 – 0.28 = 0.72.

Example 24. *What is the chance that a leap year selected at random will contain 53 Sundays ?*

Solution. A leap year consists of 52 complete weeks with a balance of two days. These two days can be any one of the seven possibilities :

(*i*) M & T, (*ii*) T & W, (*iii*) W & TH, (*iv*) TH & F, (*v*) F & Sat, (*vi*) Sat & S, (*vii*) S & M.

Out of these 7 combinations, only last two (having sundays) are favourable.

$\therefore$ Required chance = 2/7.

Example 25. *An urn contains 7 red and 4 blue balls. Two balls are drawn at random with replacement. Find the probability of getting :* (a) 2 *red balls,* (b) 2 *blue balls,* (c) *one red and one blue balls.*

Solution. (a) P (2 red balls) $= \dfrac{7}{11} \times \dfrac{7}{11} = \dfrac{49}{121}$ (b) P (2 blue balls) $= \dfrac{4}{11} \times \dfrac{4}{11} = \dfrac{16}{121}$

(c) P (one red and one blue balls) $= \dfrac{7}{11} \times \dfrac{4}{11} + \dfrac{4}{11} \times \dfrac{7}{11} = \dfrac{28}{121} + \dfrac{28}{121} = \dfrac{56}{121}$.

Example 26. *A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that—*

(*i*) *only one of them will be selected* ? (*ii*) *both of them will be selected* ? (*iii*) *none of them will be selected* ?

Solution. (*i*) P (only one of them will be selected)= $P(H).P(\overline{W}) + P(W).P(\overline{H}) = \frac{1}{7}\times\frac{4}{5}+\frac{1}{5}\times\frac{6}{7}=\frac{10}{35}=\frac{2}{7}$.

(*ii*) P (both of them will be selected)= $P(H).P(W) = \frac{1}{7}\times\frac{1}{5}=\frac{1}{35}$

(*iii*) P (none of them will be selected) = $P(\overline{H}).P(\overline{W}) = \frac{4}{5}\times\frac{6}{7}=\frac{24}{35}$.

Example 27. *Discuss and criticise the following* : $P(A) = 2/3$, $P(B) = 1/4$, $P(C) = 1/6$ *where A, B and C are mutually exclusive events.*

Solution. *Since three events A, B and C are mutually exclusive,*

$$\therefore \quad P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) = \frac{2}{3}+\frac{1}{4}+\frac{1}{6}=\frac{13}{12} > 1.$$

Since the sum of all probabilities can not be greater than unity, therefore, the given statement is wrong.

Example 28. *A coin is tossed successively three times. Determine the probability of getting* : (a) *exactly* 2 *heads,* (b) *at least* 2 *heads,* (c) *at most* 2 *heads.*

Solution. $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT.\}$

(a) P (exactly 2 heads) = $\frac{3}{8}$.

(b) P (at least 2 heads) = P (exactly 2 heads) + P (exactly 3 heads) = $\frac{3}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$.

(c) P (at most 2 heads) = P (no head) + P (1 head) + P (2 heads) = $\frac{1}{8}+\frac{3}{8}+\frac{3}{8}=\frac{7}{8}$.

Alternatively, P (at most 2 heads) = $1 - P$ (exactly 3 heads) = $1-\frac{1}{8}=\frac{7}{8}$.

Example 29. *A problem in statistics is given to students whose chance of solving it is 1/2, 1/3 and 1/4 respectively. What is the probability that the problem will be solved ?*

Solution. Here $p_1 = 1/2, p_2 = 1/3, p_3 = 1/4$. The problem will be solved if at least one student solves it.

$\therefore$ Reqd. prob. = $1 - (1 - 1/2)(1 - 1/3)(1 - 1/4) = 1 - \frac{1}{2}\times\frac{2}{3}\times\frac{3}{4} = 1 - \frac{1}{4} = \frac{3}{4}$.

Example 30. *In two successive throws of two dice, determine the probability of getting a total of 8, each time.*

Solution. Two dice can be thrown in $6 \times 6 = 36$ ways.

A total of 8 can be obtained as (6, 2), (2, 6), (5, 3), (3, 5), (4, 4), i.e.., in 5 ways.

Since P (a total of 8) = $\frac{5}{36}$, therefore reqd. probability = $\frac{5}{36}\times\frac{5}{36}=\frac{25}{1296}$.

Example 31. *In a single throw of two dice, find the probability of getting a doublet.*

Solution. Two dice can be thrown simultaneously in $6 \times 6 = 36$ ways. A doublet can be obtained as

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), *i.e.*, in 6 ways.

$\therefore$ Reqd. Prob. = $6/36 = 1/6$.

Example 32. *A die is thrown* 3 *times. Getting a '5 or 6' is considered a success. Find the probabilty of getting* : (a) 3 *successes,* (b)exactly 2 *successes,* (c) *at most* 2 *successes,* (d) *at least* 2 *successes.*

Solution. Here 'p' $= \frac{2}{6}=\frac{1}{3}$ and 'q' $= 1-\frac{1}{3}=\frac{2}{3}$ (prob. of failure)

(a) P (3 successes) = $\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{1}{27}$.

(b) P (exactly 2 successes) = $\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{2}{3}\times\frac{1}{3}+\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{2}{27}+\frac{2}{27}+\frac{2}{27}=\frac{6}{27}=\frac{2}{9}$.

(c) P (at most 2 successes)= $1 - P$ (3 successes) = $1-\frac{1}{27}=\frac{27-1}{27}=\frac{26}{27}$.

(d) P (at least 2 successes)= P (exactly 2 successes) + P (3 successes) = $\frac{2}{9}+\frac{1}{27}=\frac{6+1}{27}=\frac{7}{27}$.

Example 33. *In a single throw of two dice, find the prob. that neither a doublet (same number on both the dice) nor total of 9 will appear.*

Solution. Two dice can be thrown in $6 \times 6 = 36$ ways.

A doublet can be obtained as (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), *i.e.*, in 6 ways

A total of 9 can be obtained as (6, 3), (3, 6), (5, 4), (4, 5), *i.e.*, in 4 ways.

$\therefore$ Number of favourable ways of getting neither a doublet nor a total of 9 $= 36 - (6 + 4) = 26$ ways.

$\therefore$ Reqd. Prob. $= \dfrac{\text{no. of favourable ways}}{\text{total no. of ways}} = \dfrac{26}{36} = \dfrac{13}{18}$.

Example 34. *Two dice are thrown. Find the probability that a multiple of 2 occurs on one die and a multiple of 3 occurs on the other.*

Solution. Two dice can be thrown in $6 \times 6 = 36$ ways. Multiples of 2 are 2, 4, 6 and multiples of 3 are 3, 6.

Favourable cases for getting multiple of 2 on one die and multiple of 3 on the other are :

(2, 3), (2, 6), (4, 3), (4, 6), (6, 3), (6, 6), (6, 4), (6, 2), (3, 2), (3, 4) and (3, 6), *i.e.*, 11 ways.

$\therefore$ Reqd. prob. $= 11/36$.

Example 35. *An urn contains 25 balls numbered 1 through 25. Suppose an odd number is considered a 'success'. Two balls are drawn from the urn with replacement. Find the prob. of getting :*

(a) *two successes*, (b) *exactly one success*, (c) *at least one success*, (d) *no success.*

Solution. Total no. of balls in the urn = 25. Number of balls bearing odd numbers = 13.

Let 'p' denote the prob. of success and 'q' the prob. of failure. Therefore, $p = 13/25$ and $q = 12/25$.

(*a*) P (two successes) $= p \times p = \dfrac{13}{25} \times \dfrac{13}{25} = \dfrac{169}{625}$.

(*b*) P (exactly one success) $= p \times q + q \times p = \dfrac{13}{25} \times \dfrac{12}{25} + \dfrac{12}{25} \times \dfrac{13}{25} = \dfrac{156}{625} + \dfrac{156}{625} = \dfrac{312}{625}$.

(*c*) P (at least one success) $= P$ (exactly one) $+ P$ (two successes) $= \dfrac{312}{625} + \dfrac{169}{625} = \dfrac{481}{625}$.

(*d*) P (no success) $= q \times q = \dfrac{12}{25} \times \dfrac{12}{25} = \dfrac{144}{625}$.

Example 36. *Arun and Tarun appear for an interview for two vacancies. The probability of Arun's selection is 1/3 and that of Tarun's selection is 1/5. Find the probability that one of them will be selected.*

Solution. P (Arun's selection) $= 1/3$, $\quad P$ (Arun's not selection) $= 1 - \dfrac{1}{3} = \dfrac{2}{3}$.

P (Tarun's selection) $= 1/5$., P (Tarun's not selection) $= 1 - \dfrac{1}{5} = \dfrac{4}{5}$.

$\therefore$ Reqd. prob. $= \dfrac{1}{3} \times \dfrac{4}{5} + \dfrac{1}{5} \times \dfrac{2}{3} = \dfrac{4}{15} + \dfrac{2}{15} = \dfrac{6}{15} = \dfrac{2}{5}$.

Example 37. *What is the probability of throwing at least two sevens in 3 throws of a pair of dice ?*

Solution. Two dice simultaneously can be thrown in $= 6 \times 6 = 36$ ways.

A total of 7 can be obtained as (6, 1), (1, 6), (5, 2), (2, 5), (4, 3), (3, 4) = 6 ways.

$\therefore$ In a single throw of a pair of dice, the prob . of getting a total of 7 *i.e.*, $p = 6/36 = 1/6$.

Reqd. Prob. $= \dfrac{1}{6} \times \dfrac{1}{6} \times \dfrac{5}{6} + \dfrac{1}{6} \times \dfrac{5}{6} \times \dfrac{1}{6} + \dfrac{5}{6} \times \dfrac{1}{6} \times \dfrac{1}{6} + \dfrac{1}{6} \times \dfrac{1}{6} \times \dfrac{1}{6} = \dfrac{5}{216} + \dfrac{5}{216} + \dfrac{5}{216} + \dfrac{1}{216} = \dfrac{16}{216} = \dfrac{2}{27}$.

Example 38. *Three persons work independently to decipher a message in Morse Code. The respective probabilities of their deciphering the code are* $\dfrac{1}{4}, \dfrac{1}{3}$ and $\dfrac{1}{5}$. *What is the probability that the mesage will be deciphered ?*

Solution. Let the three persons be A, B and C. Then,

$$P(A) = \frac{1}{4},\ P(B) = \frac{1}{3},\ P(C) = \frac{1}{5} \cdot\ P(\bar{A}) = 1 - \frac{1}{4} = \frac{3}{4},\ P(\bar{B}) = 1 - \frac{1}{3} = \frac{2}{3},\ P(\bar{C}) = 1 - \frac{1}{5} = \frac{4}{5}.$$

$\therefore$ Reqd. prob. $= P(A)\, P(\bar{B})\, P(\bar{C}) + P(\bar{A})\, P(B)\, P(\bar{C}) + P(\bar{A})\, P(\bar{B})\, P(C)$

$$= \frac{1}{4} \times \frac{2}{3} \times \frac{4}{5} + \frac{3}{4} \times \frac{1}{3} \times \frac{4}{5} + \frac{3}{4} \times \frac{2}{3} \times \frac{1}{5} = \frac{8}{60} + \frac{12}{60} + \frac{6}{60} = \frac{26}{60} = \frac{13}{30}.$$

Example 39. *In a simultaneous toss of 4 coins, what is the probability of getting*

(a) *less than 3 heads* ? (b) *more than 3 heads* ? (c) *exactly 3 heads* ?

Solution. $S = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTTH, TTHH, THHT, THTH, HTHT, HTTT, THTT, TTHT, TTTH, TTTT\}$

(a) P (*less than 3 heads*) $= P$ (1 *head*) $+ P$ (2 *heads*) $+ P$ (*no head*) $= \frac{4}{16} + \frac{6}{16} + \frac{1}{16} = \frac{11}{16}$.

(b) P (*more than 3 heads*) $= P(HHHH) = 1/16$.

(c) P (exatly3 heads) $= 4/16$.

Example 40. *A bag contains* 13 *balls numbered from* 1 *through* 13. *Suppose an even number is considered a 'success'. Two balls are drawn, with replacement from the bag. Find the probability of getting :*

(a) *two successes,* (b) *no success,* (c) *exactly one success,* (d) *at least one success.*

Solution. Even numbers from 1 to 13 are 2, 4, 6, 8, 10, 12.

(a) P (two successes) $= \frac{6}{13} \times \frac{6}{13} = \frac{36}{169}$ (b) P (no success) $= \frac{7}{13} \times \frac{7}{13} = \frac{49}{169}$

(c) P (exactly one success) $= \frac{6}{13} \times \frac{7}{13} + \frac{7}{13} \times \frac{6}{13} = \frac{42}{169} + \frac{42}{169} = \frac{84}{169}$

(d) P (at least one success) $= 1 - P$(no success) $= 1 - \frac{7}{13} \times \frac{7}{13} = 1 - \frac{49}{169} = \frac{120}{169}$

Example 41. *Three cards are drawn from a well-shuffled deck of cards, one after the other and with replacement. What is the probability that—*

(a) *all the three cards are spades ?* (b) *first two cards are queens and third card is a black ace ?*

(c) *first card is jack, second is a red queen and third card is a king ?*

Solution. (a) P (all the three cards are spades) $= \frac{13}{52} \times \frac{13}{52} \times \frac{13}{52} = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$.

(b) Reqd. prob. $= \frac{4}{52} \times \frac{4}{52} \times \frac{2}{52} = \frac{1}{13} \times \frac{1}{13} \times \frac{1}{26} = \frac{1}{4394}$. (c) Reqd. prob. $= \frac{4}{52} \times \frac{2}{52} \times \frac{4}{52} = \frac{1}{13} \times \frac{1}{26} \times \frac{1}{13} = \frac{1}{4394}$.

Example 42. *A drawer contains* 50 *bolts and* 150 *nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or is a bolt ?*

Solution. Let 'A' = {the item is rusted} 'B' = {the item is a bolt}. Here A and B are non-mutually exclusive events.

$\therefore$ $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{100}{200} + \frac{50}{200} - \frac{25}{200} = \frac{150 - 25}{200} = \frac{125}{200} = \frac{5}{8}$.

$\therefore$ Prob. (the item is rusted or the item is a bolt) $= 5/8$. **Ans.**

Example 43. *In two sucessive throws of a pair of dice, determine the probability of getting a total of odd number of points each time.*

Solution Two dice can be thrown in $6 \times 6 = 36$ ways. A total of odd number of points can be obtained as : (1, 2), (2, 1), (3, 2), (2, 3), (4, 1), (1, 4), (6, 1), (1, 6), (5, 2), (2, 5), (3, 4), (4, 3), (6, 3), (3, 6), (5, 4), (4, 5), (6, 5), (5, 6) = 18 ways.

$\therefore$ Reqd. prob. $= \frac{18}{36} \times \frac{18}{36} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Example 44. *A draw two cards with replacement from a well-shuffled deck of cards and at the same time B throws a pair of dice. What is the probability that—*

(a) *A gets both cards of the same suit and B gets a total of* 6 ? (b) *A gets two jacks and B gets a doublet ?*

Solution. (a) $P(A) = 4 \times \frac{13}{52} \times \frac{13}{52} = 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{4}$, and $P(B) = 5/36$.

[$\because$ a total of 6 can be obtained in 5 ways as (5, 1), (1, 5), (4, 2), (2, 4), (3, 3,)]

$\therefore$ Reqd. prob. $= P(A) \cdot P(B) = \frac{1}{4} \times \frac{5}{36} = \frac{5}{144}$.

(b) $P(A) = \frac{4}{52} \times \frac{4}{52} = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$, and $P(B) = 6/36 = 1/6$.

$\therefore$ Reqd. prob. $= P(A) \cdot P(B) = \frac{1}{169} \times \frac{1}{6} = \frac{1}{1014}$.

Example 45. *Show that prob. of an impossible event is zero.*

Solution. Impossible event contains no sample points and therefore, a certain event A and impossible event ϕ are mutually exclusive. Therefore, $A \cup \phi = A \Rightarrow$ $P(A \cup \phi) = P(A) \Rightarrow$ $P(A) + P(\phi) = P(A) \Rightarrow$ $P(\phi) = 0$.

Example 46. *A book contains* 1000 *pages. A page is chosen at random. What is the chance that the sum of digits on a page is equal to* 9.

Solution. A page can be chosen from 1000 pages in 1000 ways.

Number of favourable ways of getting a page having the sum of digits on it 9 is = 55.

viz. {9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 108, 117, 126, 135, 144, 153, 162, 171, 180, 207, 216, 225, 234, 243, 252, 261, 270, 306, 315, 324, 333, 342, 351, 360, 405, 414, 423, 432, 441, 450, 504, 513, 522, 531, 540, 603, 612, 621, 630, 702, 711, 720, 801, 810, 900,}.

$\therefore$ Reqd. prob. = 55/1000 = 11/200.

Example 47. *A coin is tossed three times. Find the chance that head and tail will show alternately.*

Solution. Let (H) and (T) denote the occurrence of head and tail in a toss respectively.

There can be two possibilities : (H) (T) (H) and (T) (H) (T).

Prob. of either possibility $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$. [$S$ = {HHH, HHT, ..., 8 elements}

But these are mutually exclusive events. Prob. of each sample point $= \frac{1}{8}$

Hence reqd. prob. $= \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$. $\therefore$ Required prob. $= \frac{2}{8}$.]

Example 48. *In shuffling a pack of cards, four are accidentally dropped. Find the chance that missing cards should be one from each suit.*

Solution. No. of fav. ways of getting four cards, one from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$.

Total ways of getting 4 cards from a pack of 52 cards $= {}^{52}C_4$.

$$\therefore \quad \text{Reqd. prob.} = \frac{\text{no. of fav. ways}}{\text{total no. of ways}} = \frac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4} = \frac{13 \times 13 \times 13 \times 13}{52 \times 51 \times 50 \times 49} \times 4 \times 3 \times 2 \times 1 = \frac{2197}{20825}.$$

Example 49. *From 20 tickets marked with the first 20 numerals, one is drawn at random. Find the chance that it is a multiple of 3 or 7.*

Solution. Let 'A' = {*a number multiple of* 3} and 'B' = {*a number multiple of* 7}.

Numbers which are multiple of 3 are 3, 6, 9, 12, 15, 18 and numbers which are multiple of 7 are 7, 14.

Here A and B are mutually exclusive events.

$\therefore$ Reqd. chance $= P(A) + P(B) = \frac{6}{20} + \frac{2}{20} = \frac{8}{20} = \frac{2}{5}$.

Example 50. *Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, 1 girl and 3 boys. One child is selected at random from each group. Show the chance that three selected consisting of 1 girl and 2 boys is* 13/32.

Solution.

1st group	2nd Group	3rd Group
$3G + 1B$	$2G + 2B$	$1G + 3B$.

1 girl and 2 boys can be selected in any of the following three mutually exclusive ways.

$G - B - B$, $\quad B - G - B$, $\quad B - B - G$

$p_1 = \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{9}{32}$. $\quad p_2 = \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{3}{32}$. $\quad p_3 = \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{1}{32}$.

$\therefore$ Required chance $= \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}$.

Example 51. *Suppose it is 9 to 7 against a person A who is now 35 years of age living till he is 65 and 3 : 2 against a person B now 45 living till he is 75. Find the chance that one at least of these persons will be alive 30 years hence.*

Solution. The chances that A and B will die within 30 years are 9/16 and 3/5, respectively.

The events are independent, therefore, the chance that both will die is $= \frac{9}{16} \times \frac{3}{5} = \frac{27}{80}$.

$\therefore$ The chance that at least one will be alive is $= 1 - \frac{27}{80} = \frac{53}{80}$.

Example 52. *Find the probability of drawing two spades from a well-shuffled deck of 52 cards if (i) the first card is replaced before the second one is taken (ii) the first one is not replaced.*

Solution.

(i) Reqd. probability $= \frac{^{13}C_1}{^{52}C_1} \times \frac{^{13}C_1}{^{52}C_1} = \frac{13}{52} \times \frac{13}{52} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$.

(ii) Reqd. probability $= \frac{^{13}C_1}{^{52}C_1} \times \frac{^{12}C_1}{^{51}C_1} = \frac{13}{52} \times \frac{12}{51} = \frac{1}{4} \times \frac{4}{17} = \frac{1}{17}$.

Example 53. *An integer is chosen at random from first two hundred digits. What is the prob. that the integer chosen is divisible by 6 or 8 ?*

Solution Let 'A' = {integer chosen is divisible by 6}, and 'B' = {integer chosen is divisible by 8}.

Therefore, $P(A) = 33/200$, $P(B) = 25/200$, $P(A \cap B) = 8/200$.

Since A and B are not mutually exclusive, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{58-8}{200} = \frac{1}{4}.$$

$\therefore$ The prob. that integer chosen is divisible by 6 or 8 = 1/4.

Example 54. *If for two mutually exclusive events A and B, P(B) = 2P(A) and $A \cup B = S$, find P(A).*

Solution. Given $A \cup B = S$, $P(A \cup B) = P(S) = 1$.

But A and B are mutually exclusive events. Therefore, $P(A \cup B) = P(A) + P(B)$

$\therefore$ $P(A) + P(B) = 1$ or $P(A) + 2P(A) = 1$ [$\because P(B) = 2P(A)$]

or $3P(A) = 1$ or $P(A) = 1/3$. Therefore, $P(B) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$.

Example 55. *The probability of n independent events are $p_1, p_2, ..., p_n$. Find an expression for the probability that at least one of the events will happen.*

Solution. Prob. of non-happening of 1st event $= 1 - p_1$

Prob. of non-happening of 2nd event $= 1 - p_2$

... ...

Prob. of non-happening of nth event $= 1 - p_n$.

Since the events are independent, therefore prob. of non-happening of all the events

$$= (1 - p_1)(1 - p_2) \ldots \ldots (1 - p_n).$$

$\therefore$ Prob. of happening of at least one of the events $= 1 - (1 - p_1)(1 - p_2) \ldots \ldots (1 - p_n)$.

Example 56. *Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the balls drawn not being replaced.*

Solution. Let A = event that ball drawn is white in Ist draw, B = event that ball drawn is white in 2nd draw.

But $P(A \cap B) = P(A) \cdot P(B \mid A)$. Here $P(A) = 7/12$, $P(B \mid A) = 6/11$.

Therefore, $P(A \cap B) = \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$

Exmple 57. *The odds against a certain event are 5 to 2 and the odds in favour of another (independent) event are 6 to 5. Find the chance that at least one of the events will happen.*

Solution. Prob. that first event will not happen $= \frac{5}{5+2} = \frac{5}{7}$.

The prob. that second event will not happen $= 1 - \frac{6}{6+5} = \frac{5}{11}$.

$\therefore$ Prob. that none of the events will happen $= \frac{5}{7} \times \frac{5}{11} = \frac{25}{77}$ (since the events are independent)

Therefore, the prob. that at least one of the events will happen $= 1 - \frac{25}{77} = \frac{52}{77}$.

Example 58. *'A' speaks truth in 60% cases and 'B' in 70% cases. In what percentage of cases are they likely to contradict each other in stating the same fact ?*

Solution. It is clear that they will contradict each other only if one of them speaks the truth and other speaks a lie. The prob. that A speaks the truth and B a lie $= \frac{60}{100} \times \frac{30}{100} = \frac{9}{50}$.

The prob. that B speaks the truth and A a lie $= \frac{70}{100} \times \frac{40}{100} = \frac{14}{50}$.

$\therefore$ Reqd. prob. $= \frac{9}{50} + \frac{14}{50} = \frac{23}{50}$. $\therefore$ The % of cases in which they contradict each other $= 23/50 \times 100 = 46\%$.

Example 59. *In a hand at whrist what is the chance that the four kings are held by a specified player ?*

Solution. Total no. of ways in which the hand can be made up is ${}^{52}C_{13}$.

The no. of fav. ways is the same as the no. of ways in which the 9 other cards forming the hand can be chosen from the remaining 48 cards, which is ${}^{48}C_9$.

$\therefore$ Reqd. chance $= \dfrac{{}^{48}C_9}{{}^{52}C_{13}} = \dfrac{11}{4165}$.

Example 60. *A coin is tossed until a head appears or until it has been tossed three times. Given that head does not occur on the first toss, what is the prob. that coin is tossed three times ?*

Solution. Here $S = \{H, TH, TTH, TTT\}$ with associated probabilities :

$$P(H) = 1/3,\ P(TH) = 1/4,\ P(TTH) = 1/8,\ P(TTT) = 1/8.$$

Let B be the given event "no head on first toss". Then $B = \{TH, TTH, TTT\}$ and $P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$.

Let A be the event "coin is tossed three times". Then $A = \{TTH, TTT\}$ and $P(A) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

We have to find $P(A \mid B)$. Since $A \cap B = A$, therefore $P(A \cap B) = P(A) = 1/4$ and $P(B) = 1/2$.

$$\therefore \qquad P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = 1/2.$$

Example 61. *A lady buys a dozen eggs of which two turn out to be bad. She choses four eggs to scramble for breakfast. Find the chance that she choses :*

(a) *all good eggs,* (b) *three good and one bad,* (c) *Two good and two bad,* (d) *at least one bad egg.*

Solution. (a) The prob. of having all good eggs $= \dfrac{{}^{10}C_4}{{}^{12}C_4} = \dfrac{210}{495} = \dfrac{14}{33}$.

(b) The prob. of having 3 good and one bad $= \dfrac{{}^{10}C_3 \times {}^{2}C_1}{{}^{12}C_4} = \dfrac{120 \times 2}{495} = \dfrac{16}{33}$.

(c) The prob. of having 2 good and 2 bad eggs $= \dfrac{{}^{10}C_2 \times {}^{2}C_2}{{}^{12}C_4} = \dfrac{45}{495} = \dfrac{1}{11}$.

(d) The prob. of having at least one bad egg $= \dfrac{16}{33} + \dfrac{1}{11} = \dfrac{19}{33}$.

Example 62. *A six faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. What is the probability that the sum of the two number thrown is even ?*

Solution Let p be the prob. of getting an even number in a single throw of a die and q be that of an odd number. Here $p = 2/3$, $q = 1/3$.

There are two mutually exclusive cases in which the event can occur :

(*i*) an odd number in the first throw and again an odd number in the second throw,

(*ii*) an even number in the first throw and again an even number in the second throw.

Since the cases are mutually exclusive, the required probability $= \frac{1}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{2}{3} = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$.

Example 63. *Urn I contains three green and five red balls. Urn II contains two green, one red and two yellow balls. We select an urn at random and then draw one ball at random from that urn. What is the probability that we obtain a green ball ?*

Solution. The event *"green ball selected"* can occur in one of these two mutually exclusive ways :

(*i*) Select urn I and draw a green ball, or (*ii*) select urn II and draw a green ball.

$\therefore$ P (green ball) = P(urn I and green ball) + P(urn II and green ball)

$$= P(\text{urn I}) \,.\, P(\text{green/urn I}) + P(\text{urn II}) \,.\, P(\text{green/urn II}) = \frac{1}{2} \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{2}{5} = \frac{31}{80}.$$

Example 64. *The chance that A can solve a certain problem is 1/4; the chance that B can solve it is 2/3. Find the chance that the problem will be solved if they both try.*

Solution. Here $p_1 = 1/4, q_1 = 3/4$, and $p_2 = 2/3,\ q_2 = 1/3$.

$\therefore$ The prob. that problem will not be solved $= \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$.

$\therefore$ The prob. that problem will be solved $= 1 - \frac{1}{4} = \frac{3}{4}$.

Example 65. *A person writes letters for his friends and addresses three envelopes. These letters are placed at random by his little son and dispatched. What is the chance that no friend receives the correct letter ?*

Solution. The total no. of ways of placing 3 letters in 3 envelops = 3 !

All the letters can be placed correctly in one way. Hence its chance $= \frac{1}{3!}$

$\therefore$ The prob. that all letters are sent wrongly $= 1 - \frac{1}{3!} = 1 - \frac{1}{6} = \frac{5}{6}$.

Example 66. *A party of 23 persons take their seats at a round table. Show that the odds are 10 to 1 against two persons sitting together.*

Solution. *A* having taken his place, *B* has 2 choice of 22 places, 2 of which are next to *A*.

$\therefore$ Reqd. prob. $= \frac{2}{22} = \frac{1}{11} = \frac{1}{1+10}$.

$\therefore$ Odds are 10 to 1 against two persons sitting together.

Example 67. *A class consists of* 80 *students,* 25 *of them are girls and* 55 *bays,* 10 *of them are rich and the remaining poor,* 20 *of them are fair complexioned. What is the prob. of selecting a fair compelxioned rich girl ?*

Solution. Prob. of selecting a fair complexioned person $= \frac{20}{80} = \frac{1}{4}$.

Prob. of selecting a rich person$= \frac{10}{80} = \frac{1}{8}$ and prob. of selecting a girl $= \frac{25}{80} = \frac{5}{16}$.

$\therefore$ The prob. of selecting a fair complexioned rich girl $= \frac{1}{4} \times \frac{1}{8} \times \frac{5}{16} = \frac{5}{512} = 0.009$.

Example 68. *A university has to select an examiner from a list of* 50 *persons.* 20 *of them are women and* 30 *men,* 10 *of them knowing Hindi and 40 not, 15 of them being teachers and the remaining* 35 *not. What is the probability of the university selecting a Hindi knowing women teacher ?*

Solution. Prob. of selecting a woman = $20/50$. Prob. of selecting a teacher = $15/50$.

Prob. of selecting a Hindi-knowing candidate = $10/50$.

Since the events are independent, the prob. of the university selecting a Hindi-knowing woman teacher

$$= \frac{20}{50} \times \frac{15}{50} \times \frac{10}{50} = \frac{3}{125}.$$

Example 69. *A can hit a target 3 times in 5 shots; B, 2 times in 5 shots; C, 3 times in 4 shots. They fire a volley. What is the prob. that two shots hit ?*

Solution. Here $p_1 = 3/5, p_2 = 2/5, p_3 = 3/4$.

The shots will hit in the following three mutually exclusive ways :

(*i*) When *A* and *B* hit and *C* fails, prob. $= \frac{3}{5} \times \frac{2}{5} \times \frac{1}{4} = \frac{6}{100}$.

(*ii*) When *B* and *C* hit and *A* fails, prob. $= \frac{2}{5} \times \frac{3}{4} \times \frac{2}{5} = \frac{12}{100}$.

(*iii*) When *C* and *A* hit *B* fails, prob. $= \frac{3}{5} \times \frac{3}{4} \times \frac{3}{5} = \frac{27}{100}$.

These are three mutually exclusive events. Therefore, prob. that any two shots should hit

$$= \frac{6}{100} + \frac{12}{100} + \frac{27}{100} = \frac{9}{20}.$$

Example 70. *How different sets of 4 cards can be selected from a pack of 52 cards ? How many different pairs of aces can be selected ? Hence show that if two cards are selected at random from a pack, the prob. of their being aces is* $1/221$.

Solution. Different sets of 4 cards can be selected from a pack of 52 cards in ${}^{52}C_4$ ways.

Different pairs of aces can be selected in ${}^4C_2$ ways.

Two cards can be drawn from a pack of 52 cards in ${}^{52}C_2$ ways.

Required prob., *i.e.*, prob. of their being aces is $= \frac{{}^4C_2}{{}^{52}C_2} = \frac{\frac{4!}{2!\,2!}}{\frac{52!}{2!\,50!}} = \frac{4! \times 50!}{2! \times 52!} = \frac{4 \times 3}{52 \times 51} = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$.

Example 71. *A card is drawn from a well-shuffled deck of 52 cards. The outcome is noted, the card is replaced and the deck is reshuffled. Another card is drawn from the deck. What is the probability that —*

(*a*) *both the cards are of the same suit ?* (*b*) *both the cards are aces ?*

(*c*) *the first card is a spade and the second card is a black king ?* (*d*) *both are face cards ?*

Solution. (*a*) Reqd. prob. $= 4 \times \left(\frac{13}{52} \times \frac{13}{52}\right) = 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{4}$.

(*b*) Reqd. prob. $= \frac{4}{52} \times \frac{4}{52} = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$. (*c*) Reqd. prob. $= \frac{13}{52} \times \frac{2}{52} = \frac{1}{4} \times \frac{1}{26} = \frac{1}{104}$.

(*d*) Number of face cards (Jack, Queen, King and Ace)= 16. $\therefore$ Reqd. prob. $= \frac{16}{52} \times \frac{16}{52} = \frac{4}{13} \times \frac{4}{13} = \frac{16}{169}$.

14.14. DISCRETE AND CONTINUOUS VARIABLES

Suppose we collect data for the size of families in a certain town. It is obvious that the number of members in each family would be in whole numbers (irrespective of age or sex). Thus, for example, there would be no family with 1.5 members. Such type of variable (number of family members in this example) is called a *discrete variable*. The school enrolment figures, number of passengers, etc., are other examples of it.

Having defined the discrete variable, we must next define the continuous type of variable. Suppose we are interested in measuring the heights of a large number of plants and if our unit of measurement is very fine, there would be no point along the scale of measurement (between the extreme values of the heights) at which we may not find the height of plant, no matter how finely we divide the scale. *A variable (the height in this example) which takes all possible values between its limits, say a and b, is known as a continuous variable.* Weights of school children, barometric pressures, temperatures etc., are other examples of it.

14.15. RANDOM VARIABLE (OR CHANCE VARIABLE) [IGNOU 98]

To define a *random variable,* we consider some experiment whose sample space is $S = \{e_1, e_2, e_3, ..., e_n\}$.

Definition. *A random variable X is a rule which associates uniquely a real number with every elementary event $e_i \in S$, $i = 1, 2, 3, ..., n$, i.e.,* ***a random variable is a real valued function which maps the sample space on to the real line.***

In other words, *a random variable is a real valued function defined over the sample space of an experiment i.e. a variable whose value is a number determined by the sample point (out-come of the experiment) of a sample space is called a random variable.*

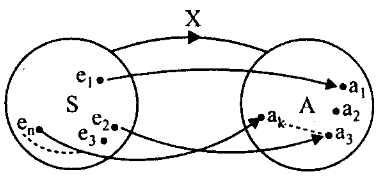

Fig. 14.2.

For example, let X be a random variable which is the number of heads obtained in two independent tosses of a fair coin. Here $S = \{HH, HT, TH, TT\}$.

Then, $X(HH) = 2$, $X(HT) = 1$, $X(TH) = 1$, $X(TT) = 0$. Therefore, X can take values 0, 1, 2.

A random variable is also known as *stochastic variable.*

As another example, let us consider a three coin throwing experiment. If we write '*H*' for turning head up and '*T*' for turning tail up, obviously the sample space for this experiment will be

$$S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}.$$

Now, suppose that the underlying rule **h** is to *count the number of 'heads' that can turn up.* Then **h** will be a *random variable* which associates uniquely a real number with every elementary event of S as shown in the Venn diagram (see Fig. 14.3).

Similarly, we can demonstrate the random variable **t** which is a rule : *to count the number of 'tails'* that can turn up in this experiment.

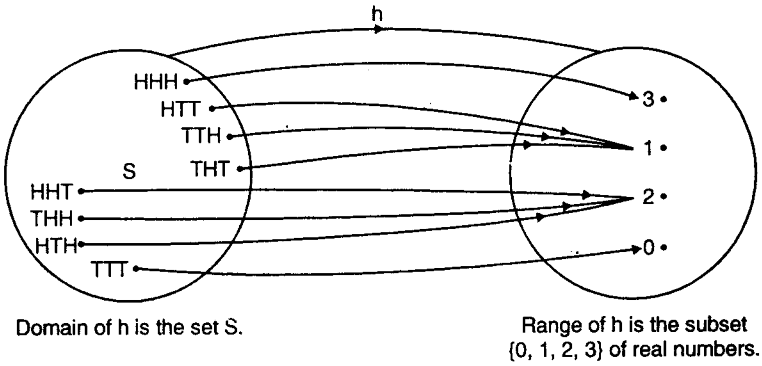

Fig. 14.3.

Furthermore, if the range of a random variable is a discrete variable, it is called a *discrete random variable.* In above example, **h** and **t** are discrete type of random variables. On the other hand, the range of *continuous random variable* will be the set of continuous real numbers. For example, if the experiment is concerned with measuring failures of an electronic component, the outcomes in this case are given by the time-to-failure which may assume any non-negative real value. In this case, the random variable is continuous.

14.16. PROBABILITY DISTRIBUTION OF A RANDOM VARIABLE

Let a random variable X assume values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$. Then different values of a random variable together with their corresponding probabilities form a probability distribution.

A random variable is said to be ***discrete*** if it assumes only a finite or infinite but denumerable number of values.

A random variable is said to be ***continuous*** if it assumes any value in an interval.

14.16-1. Discrete Probability Distribution

We consider an experiment with sample sapce $S = \{e_1, e_2, ..., e_n\}$, and suppose that a random variable X has been defined on S by the formula $X(e_i) = x_i$, $i = 1, 2, 3, ..., n$, where x_i is the real number associated with the elementary event e_i by the random variable X. It should be noted that all x_i, $i = 1, 2, ..., n$ are not necessarily different.

Now, we proceed to distribute probabilities to each event $E(x) = \{e_i \in S : X(e_i) = x_i\}$, $i = 1, 2, 3, ..., n$ *i.e.*, a subset of sample space S whose elements are *uniquely* associated to some real number, say x, by random variable X. To do this, we again associate each real number $x_1, x_2, \ldots, x_r$ $(r \le n)$ to other real numbers $P(x_1), P(x_2), ..., P(x_r)$ respectively. Then such function P is called the *probability distribution function* of random variable X and the numbers $P(x_1), P(x_2), ..., P(x_r)$ are called *probabilities* of events $E(x_1), E(x_2), \ldots, E(x_r)$ respectively, provided :

(*i*) $0 \le P(x_j) \le 1$, (*ii*) $\Sigma P(x_i) = 1$ (*i.e.* sum of all probabilities is unity), $j = 1, 2, 3, ..., r$ and $r \le n$.

Thus the probability distribution for discrete random variable X can be more conveniently represented in the following tabular form :

Table 14.1.

x	x_1	x_2	...	x_r
$P(x)$	$P(x_1)$	$P(x_2)$	...	$P(x_r)$
$P\{E(x)\}$	$P\{E(x_1)\} = p_1$	$P\{E(x_2)\} = p_2$	...	$P\{E(x_r)\} = p_r$

The following illustrative example will make the ideas clear.

Example 72. *We consider a three coin tossing experiment whose sample space is :*

$$S = \{HHH, HHT, HTH, THH, TTH, TTT, THT, HTT\}.$$

The random variable **h**, *which counts the number of heads that turn up, has the range {0, 1, 2, 3}, (see examples on p 483–485) and the four possible events are :*

(*i*) The event which no head turns up $= E(0) = \{TTT\}$

(*ii*) The event when only one head turns up $= E(1) = \{HTT, THT, TTH\}$

(*iii*) The event when only two heads turn up $= E(2) = \{HHT, HTH, THH\}$

(*iv*) The event when all the three heads turn up $= E(3) = \{HHH\}$

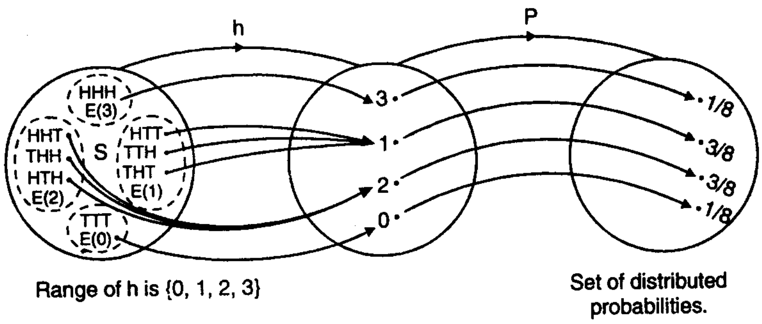

Fig. 14.4.

Thus the probability distribution is shown in Table 14.2 :

Table 14.2.

x	0	1	2	3
$P(x)$	1/8	3/8	3/8	1/8
$P\{E(x)\}$	$P\{E(0)\}$	$P\{E(1)\}$	$P\{E(2)\}$	$P\{E(3)\}$

Example 73. *Find the probability distribution of boys and girls in families with 3 children, assuming equal probabilities for boys and girls.* **[Rohilkhand 80]**

Solution. The probability of child being a boy or girl is equal to ½.

The following four possibilities may arise :

(*i*) All the three children are boys (B), $\therefore$ $P(BBB) = P(B).P(B).P(B) = ½.½.½. = ⅛$.

(*ii*) Two are boys (B) and one is a girl (G),

$\therefore$ $P(BBG + BGB + GBB) = P(BBG) + P(BGB) + P(GBB) = ½.½.½. + ½.½.½ + ½.½.½ = ⅜$.

(*iii*) Two are girls (G) and one is a boy (B), $\therefore$ $P(GGB + GBG + BGG) = ⅜$.

(*iv*) All the three are girls (G), *i.e.* no boy. $\therefore$ $P(GGG) = ½. ½. ½. = ⅛$.

Thus the probability distribution is as shown below :

No. of boys x :	0	1	2	3
Prob. $P(x)$:	⅛	⅜	⅜	⅛

14.16-2. Continuous Probability Distribution

Since X is a continuous random variable, it will have the infinite number of values in any class interval, however small. Thus, we can assign a probability to an interval of X.

The probability, that the continuous random variable X lies in the infinitesimal interval $(x, x + dx)$, can be interpreted as $f(x)\,dx$. Then $f(x)$ is known as the *probability density function* of random variable X.

The *distribution function, F(a),* usually be written as

$$F(a) = P\{X(e) \le a\} = \int_{-\infty}^{a} f(x)\,dx, \qquad ...(14.12)$$

where $P\{X(e) \le a\}$ is the probability of an event whose, each outcome e is uniquely associated with all real numbers *not more than a* by continuous random variable X. Whenever there is no ambiguity, it can be written as $P(X \le a)$.

A knowledge of the density function enables us to calculate all sorts of probabilities.

$$P(a < X \le b) = F(b) - F(a) = \int_a^b f(x)\, dx.$$

Since a and b are two extreme values on the range of X with $a < b$, we can express $P(a < X \le b)$ in terms of '*distributionfunction*' provided the distribution function $F(x)$ is differentiable, *i.e.*,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\int_a^b f(x)\, dx\right]. \qquad \text{...(14.13)}$$

For $\{X \le b\}$ is the disjoint union of two events $\{X \le a\}$ and $\{a < X \le b\}$, hence

$$P\{X \le b\} = P\{X \le a\} + P\{a < X \le b\} \qquad \text{...(14.14)}$$

by using, $P(E \cup F) = P(E) + P(F)$, for disjoint events E and F.

Using (14.12), eqn. (14.14) can be written as :

$$F(b) = F(a) + P\{a < X \le b\}$$

or

$$P\{a < X \le b\} = F(b) - F(a) = [F(x)]_a^b = \int_a^b f(x)\, dx. \quad \text{[from (14.13)]}$$

Moreover, it is evident that the area under the density function between a and b is just $P\{a \le X \le b\}$. We note that the area under the curve is zero (when $a = b$) and hence $P\{a \le X \le b\} = P\{X = a\} = 0$.

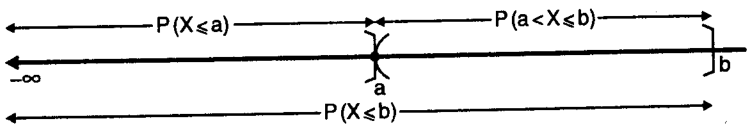

Fig. 14.5.

Therefore, for continuous random variables, and any a and b,

$$P\{a \le X \le b\} = P\{a < X \le b\} = P\{a \le X < b\}$$
$$= P\{a < X < b\} = \int_a^b f(x)\, dx = F(b) - F(a).$$

But, this is not true in the case of discrete random variables.

Some immediate consequences for any random variable X with density function $f(x)$ are :

(*i*) $\int_{-\infty}^{\infty} f(x)\, dx = 1$, (*ii*) $f(x)$ is non-negative, (*iii*) $P\{X = a\} = 0$, as pointed out earlier ....(14.15, 14.16, 14.17)

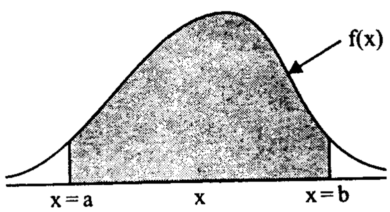

Fig. 14.6.

14.16-3. Illustrative Examples

Example 74. *Find the probability distribution of the number of sizes in three tosses of a die.*

Solution. Let 'X' denote a random variable which is the no. of sizes obtained in three tosses of a die. Clearly X can take values 0, 1, 2, 3. Here $p = 1/6$, $q = 5/6$.

$$\therefore \quad P(X = 0) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}, \quad P(X = 3) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216},$$

$$P(X = 1) = \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216} + \frac{25}{216} + \frac{25}{216} = \frac{75}{216},$$

$$P(X = 2) = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{15}{216},$$

$$P(X = 3) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}.$$

$\therefore$ Probability distribution table is

X_i	:	0	1	2	3
$P(X_i)$	:	125/216	75/216	15/216	1/216

Example 75. *Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as a number greater than 4.*

Solution. Let 'X' denote a random variable which is the no. of successes (getting a no. > 4) obtained in two tosses of a die. Clearly X takes the values 0, 1, 2. Here $p = \frac{2}{6} = \frac{1}{3}, q = 1 - \frac{1}{3} = \frac{2}{3}$.

$\therefore \quad P(X=0) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}, \; P(X=1) = \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$ and $P(X=2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$\therefore$ Probability distribution table is :

X_i	0	1	2
$P(X_i)$	$4/9$	$4/9$	$1/9$

Example 76. *Three cards are drawn successively with replacement from a well-shuffled deck of 52 cards. A random variable X denotes the no. of spades in the three cards. Determine the probability distribution of X.*

Solution. Here X can take values 0, 1, 2, and 3 : $p = \frac{13}{52} = \frac{1}{4}, \quad q = 1 - \frac{1}{4} = \frac{3}{4}$.

$$\therefore \quad P(X=0) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}, \; P(X=1) = 3 \times \left(\frac{1}{4} \times \frac{3}{4} \times \frac{3}{4}\right) = \frac{27}{64},$$

$$P(X=2) = 3 \times \left(\frac{1}{4} \times \frac{1}{4} \times \frac{3}{4}\right) = \frac{9}{64}, \; P(X=3) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}.$$

$\therefore$ Probability distribution table is :

X_i	0	1	2	3
$P(X_i)$	$27/64$	$27/64$	$9/64$	$1/64$

Example 77. *An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in three draws, with replacement, from the urn.*

Solution. Total no. of balls = 4 + 3 = 7. Here $p = 3/7, \; q = 4/7$.

Let X denote a random variable which is the no. of red balls obtained in three draws.

Clearly X can take values 0, 1, 2, 3.

$$\therefore \quad P(X=0) = \frac{4}{7} \times \frac{4}{7} \times \frac{4}{7} = \frac{64}{343}, \; P(X=1) = 3 \times \left(\frac{3}{7} \times \frac{4}{7} \times \frac{4}{7}\right) = \frac{144}{343},$$

$$P(X=2) = 3 \times \left(\frac{3}{7} \times \frac{3}{7} \times \frac{4}{7}\right) = \frac{108}{343}, \; P(X=3) = \frac{3}{7} \times \frac{3}{7} \times \frac{3}{7} = \frac{27}{343}.$$

$\therefore$ Probability distribution table is :

X_i	0	1	2	3
$P(X_i)$	$64/343$	$144/343$	$108/343$	$27/343$

Example 78. *Find the probability distribution of the number of doublets in four throws of a pair of dice.*

Solution. Let 'X' denote random variable which is the no. of doublets obtained in four throws of a pair of dice. Clearly X takes the values 0, 1, 2, 3, 4. $p = 6/36 = 1/6, \; q = 5/6$.

$$\therefore \quad P(X=0) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{625}{1296}, \; P(X=1) = 4 \times \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) = \frac{500}{1296}, \; P(X=4) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{1296}.$$

$$P(X=2) = 6 \times \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) = \frac{150}{1296} \cdot P(X=3) = 4 \times \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) = \frac{20}{1296}.$$

$\therefore$ Probability distribution table is :

X_i	0	1	2	3	4
$P(X_i)$	$625/1296$	$500/1296$	$150/1296$	$20/1296$	$1/1296$

Example 79. *A die is tossed twice. A 'success' is getting an odd number on each toss. Find the probability distribution of the number of successes.*

Solution. Let 'X' denote a random variable which is the no. of successes obtained in two tosses of a die. Clearly X can take values 0, 1, 2.

Now prob. of getting an odd number in a single toss of a die = $3/6 = 1/2$.

$\therefore \quad p$ (*i.e.* prob. of success) = $1/2$ and q (*i.e.* prob. of failure) = $1 - \frac{1}{2} = \frac{1}{2}$.

$\therefore$ $P(X=0) = q \times q = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, $P(X=2) = p \times p = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$,

$P(X=1) = p \times q + q \times p = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

$\therefore$ Probability distribution table is :

X_i	0	1	2
$P(X_i)$	1/4	1/2	1/4

Example 80. *Two bad eggs are mixed accidentally with 10 good ones. Find the probability distribution of the number of bad eggs in 3, drawn at random, without replacement from the lot.*

Solution. Let 'X' denote a random variable which is the number of bad eggs obtained in 3 draws.

Clearly X takes the values 0, 1 and 2 . Total number of eggs = 12, number of bad eggs = 2, number of good eggs = 10.

$\therefore$ $P(X=0) = \frac{10}{12} \times \frac{9}{11} \times \frac{8}{10} = \frac{12}{22}$.

$P(X=1) = \left(\frac{2}{12} \times \frac{10}{11} \times \frac{9}{10}\right) + \left(\frac{10}{12} \times \frac{2}{11} \times \frac{9}{10}\right) + \left(\frac{10}{12} \times \frac{9}{11} \times \frac{2}{10}\right) = \frac{3}{22} + \frac{3}{22} + \frac{3}{22} = \frac{9}{22}$.

$P(X=2) = \left(\frac{2}{12} \times \frac{1}{11} \times \frac{10}{10}\right) + \left(\frac{2}{12} \times \frac{10}{11} \times \frac{1}{10}\right) + \left(\frac{10}{12} \times \frac{2}{11} \times \frac{1}{10}\right) = \frac{1}{66} + \frac{1}{66} + \frac{1}{66} = \frac{3}{66} = \frac{1}{22}$.

$\therefore$ Probability distribution table is :

X_i	0	1	2
$P(X_i)$	12/22	9/22	1/22

Example 81. *Two cards are drawn without replacement from a well-shuffled deck. Determine the probability distribution of the number of face cards (Jack, queen, king, and ace).*

Solution. Let 'X' denote a random variable which is the no. of face cards obtained in two draws. Clearly X takes the values 0, 1, 2. Here $p = \frac{16}{52}$, $q = \frac{36}{52}$.

$\therefore$ $P(X=0) = \frac{36}{52} \times \frac{35}{51} = \frac{105}{221}$, $P(X=1) = \frac{16}{52} \times \frac{36}{51} + \frac{36}{52} \times \frac{16}{51} = \frac{48}{221} + \frac{48}{221} = \frac{96}{221}$, $P(X=2) = \frac{16}{52} \times \frac{15}{51} = \frac{20}{221}$.

$\therefore$ Probability distribution table is :

X_i	0	1	2
$P(X_i)$	105/221	96/221	20/221

14.17. MEAN, VARIANCE AND STANDARD DEVIATION (S.D.)

If the random variable X assumes the discrete values $x_1, x_2, x_3, ..., x_r$ with corresponding probabilities $p_1, p_2, p_3, \ldots, p_r$ then,

$$\textbf{Mean (expected value)} = \bar{x} = \left[\sum_{i=1}^{r} p_i x_i / \sum_{i=1}^{r} p_i\right] = \sum_{i=1}^{r} p_i x_i \qquad (\because \ \Sigma p_i = 1) \qquad ...(14.18)$$

$$\textbf{Var }(x) = \sigma_x^2 = \sum_{i=1}^{r} (x_i - \bar{x})^2 p_i = \sum_{i=1}^{r} x_i^2 p_i - (\bar{x})^2 \ \textbf{(when } \bar{x} \textbf{ is in fraction)} \qquad ...(14.19)$$

$$\textbf{S.D.} \quad = \sigma_x = \sqrt{(\textbf{Var})}\,. \qquad ...(14.20)$$

In the case of continuous random variable X with probability density function $f(x)$, we have

$$\textbf{Mean} = \bar{x} = \int_{-\infty}^{\infty} x f(x)\, dx. \qquad ...(14.21)$$

$$\textbf{Var }(x) = \sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)\, dx \ (\textit{definition}) \qquad ...(14.22a)$$

$$= \int_{-\infty}^{\infty} x^2 f(x) dx - (\bar{x})^2 \ (\textit{in computation}) \qquad ...(14.22b)$$

$$\textbf{S.D.} = \sigma_x = \sqrt{(\textbf{Var.})} \qquad ...(14.23)$$

14.17-1. Illustrative Examples

Example 82. *During the course of a day, a machine turns out either 0, 1 or 2 defective pens with probabilities 1/6, 2/3 and 1/6, respectively. Calculate the mean value and the variance of the defective pens produced by the machine in a day.*

Solution. The probability distribution in this example is given by : $P(0) = 1/6$, $P(1) = 2/3$, $P(2) = 1/6$.

$\therefore$ Mean value $= \bar{x} = \sum_{i=0}^{2} p_i x_i = 1/6 . 0 + 2/3 . 1 + 1/6 . 2 = 1.$ **Ans.**

$\text{Var}(x) = \sum_{i=0}^{2} x_i^2 p_i - (\bar{x})^2 = 1/6 . 0^2 + 2/3 . 1^2 + 1/6 . 2^2 - 1^2 = 1/3.$ **Ans.**

Example 83. *Given the following probability distribution :*

x	:	0	1	2	3	4	5	6	7
$P(x)$	:	0	2λ	2λ	λ	3λ	λ^2	$2\lambda^2$	$7\lambda^2+\lambda$

(*i*) *Find* λ, (*ii*) *Evaluate* $P(x \geq 5)$ *and* $P(x < 4)$.

Solution. (*i*) Since the sum of all probabilities is unity, therefore

$$0 + \lambda + 2\lambda + 2\lambda + 3\lambda + \lambda^2 + 2\lambda^2 + 7\lambda^2 + \lambda = 1$$

or $\quad 10\lambda^2 + 9\lambda - 1 = 0 \quad$ or $\quad (\lambda + 1)(10\lambda - 1) = 0$

$\therefore \quad \lambda = 1/10 \quad (\lambda \neq -1, \text{ since } 0 \leq p(x) \leq 1)$ **Ans.**

(*ii*) $P(x \geq 5) = p(5) + p(6) + p(7) = \lambda^2 + 2\lambda^2 + (7\lambda^2 + \lambda) = 10\lambda^2 + \lambda = 10\,(1/10)^2 + 1/10 = 1/5$ **Ans.**

and $\quad P(x < 4) = p(0) + p(1) + p(2) + p(3) = 0 + \lambda + 2\lambda + 2\lambda = 5\lambda = 5 \times 1/10 = 1/2 \quad (\because \lambda = 1/10)$ **Ans.**

Example 84. *The probability density function for a continuous random variable x is given by*

$$f(x) = \begin{cases} 1/2 \sin x \, dx, & 0 \leq x \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance.

Solution. By definitions (1.21) and 1.22b), we have

$$\text{Mean} = \bar{x} = \int_0^{\pi} \overset{I}{x} \, (\overset{II}{1/2 \sin x}) \, dx$$

$$= [\, x(-1/2 \cos x) - 1 . (-1/2 \sin x)]_0^{\pi} = \pi/2 \quad \text{(integrating by parts)}$$ **Ans.**

$$\text{Var}(x) = \int_0^{\pi} x^2 (1/2 \sin x) \, dx - (\bar{x})^2$$

$$= \Big[x^2 (-1/2 \cos x) - (2x) . (-1/2 \sin x) + (2)(1/2 \cos x)\Big]_0^{\pi} - \frac{\pi^2}{4} \quad \text{(again integrating by parts)}$$

$$= \left(\frac{\pi^2}{2} - 1\right) - 1 - \frac{\pi^2}{4} = \frac{\pi^2}{4} - 2.$$ **Ans.**

Example 85. *Find the mean number of heads in three tosses of a coin.*

Solution. Let '*X*' denote a random variable which is the no. of heads obtained in three tosses of a coin. Clearly *X* takes the values 0, 1, 2, 3. Here $p = 1/2$, $q = 1 - 1/2 = 1/2$.

$\therefore \quad P(X=0) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}, \quad P(X=1) = 3 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{8},$

$P(X=2) = 3 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{8}, \quad P(X=3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}.$

X_i :	0	1	2	3
p_i :	1/8	3/8	3/8	1/8
$p_i X_i$:	0	3/8	6/8	3/8

Mean $= \sum p_i X_i = 12/8 = 3/2$.

Example 86. *Find* μ, σ^2 *and* σ *for the probability distribution :*

Z_i	:	-3	-1	0	1	3
$P(Z_i)$	:	0.05	0.45	0.20	0.25	0.05

Solution.

Z_i	p_i	p_iZ_i	$Z_i-\mu$	$(Z_i-\mu)^2$	$(Z_i-\mu)^2p_i$
−3	0.05	−0.15	−2.8	7.84	0.3920
−1	0.45	−0.45	−0.8	0.64	0.2880
0	0.20	0.00	0.2	0.04	0.0008
1	0.25	0.25	1.2	1.44	0.3600
3	0.05	0.15	3.2	10.24	0.5120
		$\mu=-0.20$			1.5528

From this table, we have $\mu = \Sigma\, p_iZ_i = -\,0.20$

$\sigma^2 = \Sigma\,(Z_i-\mu)^2p_i = 1.5528 = 1.56$ (approx.). Therefore, $\sigma = +\sqrt{\sigma^2} = \sqrt{1.56} = 1.249$.

Example 87. *Find* μ, σ^2, and σ *for the probability distribution* :

X_i	:	1	2	3	4
$P(X_i)$	:	0.4	0.3	0.2	0.1

Solution.

X_i	p_i	p_iX_i	$X_i-\mu$	$(X_i-\mu)^2$	$(X_i-\mu)^2p_i$
1	0.4	0.4	−1	1	0.4
2	0.3	0.6	0	0	0.0
3	0.2	0.6	1	1	0.2
4	0.1	0.4	2	4	0.4
		$\mu=2$			1.0

$\mu = \Sigma p_iX_i = 2.\ \sigma^2 = \Sigma(X_i-\mu)^2p_i = 1,\ \sigma = +\sqrt{1^2} = 1.$

Example 88. *A die is tossed twice. Getting 'a number greater than 4' is considered a success. Find the variance of the probability distribution of the number of successes.*

Solution. Let 'X' denote a random variable which is the number of successes (getting a no. > 4) obtained in two tosses of a die. Probability distribution of random variable is

X_i	:	0	1	2	*(see Example 75)*
$P(X_i)$	:	$4/9$	$4/9$	$1/9$	

X_i	p_i	p_iX_i	X_i^2	$p_iX_i^2$
0	$4/9$	0	0	0
1	$4/9$	$4/9$	1	$4/9$
2	$1/9$	$2/9$	4	$4/9$
		$\Sigma\ p_iX_i = 6/9$		$\Sigma\ p_iX_i^2 = 8/9$

Thus, we get $\mu = \Sigma p_iX_i = 6/9 = 2/3$, $\sigma^2 = \Sigma\, p_iX_i^2 - \mu^2 = 8/9 - (2/3)^2 = 4/9$, $\sigma = +\sqrt{\sigma^2} = \sqrt{4/9} = 2/3$.

Example 89. *Compute the variance of the probability distribution of the number of doublets in four throws of a pair of dice.*

Solution. The probability distribution of a random variable X is

X_i	:	0	1	2	3	4	*(see Example 78)*
$P(X_i)$	:	$625/1296$	$500/1296$	$150/1296$	$20/1296$	$1/1296$	

Now, compute the following table :

X_i	p_i	p_iX_i	X_i^2	$p_iX_i^2$
0	$625/1296$	0	0	0
1	$500/1296$	$500/1296$	1	$500/1296$
2	$150/1296$	$300/1296$	4	$600/1296$
3	$20/1296$	$60/1296$	9	$180/1296$
4	$1/1296$	$4/1296$	16	$16/1296$
	$\Sigma p_i = 1$	$\Sigma\, p_iX_i = 864/1296 = 2/3$		$\Sigma\ p_iX_i^2 = 1296/1296 = 1$

Thus, we get

$\mu = \Sigma p_i X_i = 864/1296 = 2/3$. Hence, variance $\sigma^2 = \Sigma p_i X_i^2 - (\mu)^2 = 1296/1296 - (2/3)^2 = 1 - 4/9 = 5/9$.

Example 90. *Find μ, σ^2 and σ for the following probability distribution.* :

X_i	:	1	3	5	7	9
$P(X_i)$	:	0.1	0.0	0.4	0.2	0.3

Solution.

X_i	p_i	$p_i X_i$	X_i^2	$p_i X_i^2$
1	0.1	0.1	1	0.1
3	0.0	0.0	9	0.0
5	0.4	2.0	25	10.0
7	0.2	1.4	49	9.8
9	0.3	2.7	81	24.3
	$\Sigma p_i = 1$	$\Sigma p_i X_i = 6.2$		$\Sigma p_i X_i^2 = 44.2$

Thus, we get

$$\mu = \sum_{i=1}^{n} p_i X_i = 6.2,\ \sigma^2 = \Sigma p_i X_i^2 - (\mu)^2 = 44.2 - (6.2)^2 = 44.2 - 38.44 = 5.76,$$

$$\sigma = +\sqrt{\sigma^2} = \sqrt{5.76} = 2.4. \text{ Hence } \mu = 6.2,\ \sigma^2 = 5.76 \text{ and } \sigma = 2.4.$$

Example 91. *A die is tossed thrice. A 'success' is getting 1 or 6, on a toss. Find the mean and the variance of the number of successes.*

Solution. Let 'X' denote a random variable which is the number of successes (getting 1 or 6) obtained in three tosses of a die.

Here X takes the values 0, 1, 2, 3. Here $p = \frac{2}{6} = \frac{1}{3}$, $q = 1 - \frac{1}{3} = \frac{2}{3}$.

$$\therefore\quad P(X=0) = \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3} = \frac{8}{27},\ P(X=1) = 3\times\left(\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}\right) = \frac{12}{27},$$

$$P(X=2) = 3\times\left(\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}\right) = \frac{6}{27},\ P(X=3) = \frac{1}{3}\times\frac{1}{3}\times\frac{1}{3} = \frac{1}{27}.$$

Now,

X_i	p_i	$p_i X_i$	$X_i - \mu$	$(X_i - \mu)^2$	$(X_i - \mu)^2 p_i$
0	8/27	0	−1	1	8/27
1	12/27	12/27	0	0	0
2	6/27	12/27	1	1	6/27
3	1/27	3/27	2	4	4/27
		$\Sigma p_i X_i = 1$			18/27

Thus, we get $\mu = \Sigma p_i X_i = 27/27 = 1$, $\sigma^2 = \Sigma (X_i - \mu)^2 p_i = 18/27 = 2/3$. Hence mean $= 1$, variance $= 2/3$.

EXAMINATION PROBLEMS

1. A win is so weighted that head is thrice as likely to appear as tail. What is $P(H)$ and $P(T)$.
   [Ans. 3/4, 1/4.]
2. There are 26 persons in a birthday party. What is the probability that at least two of them have the same birthday ?
   [Ans. 0.60]
3. If the probability that A will solve a problem is 1/4 and the probability that B will solve it is 3/4, what is the probability that the problem is at all solved ?
   [Ans. 13/16.]
4. An urn contains 3 green and 5 red balls. One ball is drawn, its colour unnoted and laid aside. Then another ball is drawn, find the probability that it is green or red. How does the probability change if the colour of the ball is noted ?
   [Ans. 3/8, 5/8, 2/7.]
5. An urn contains a white and b black balls. Balls are drawn one by one until only those of the same colour are left. What is the probability that they are white ?
   [Ans. $a/(a+b)$]

6. A pair of fair dice is rolled once. What is the probability that the sum is equal to each of the integers from 2 to 12 ?
[Ans.

S:	2	3	4	5	6	7	8	9	10	11	12
p:	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

7. An urn contains one white and 2 black balls, while another contains 2 white and 1 black ball. One ball is transferred from the first urn into second, after which a ball is drawn from the second urn. What is the probability that it is black?
[Ans. 5/12.]

8. Three players *A*, *B* and *C* play a sequence of games. It is also decided that winner of each game scores one point and he who first scores three points is the final winner. A wins the first and third games while *B* wins the second. What is the probability that *C* is the final winner ?
[Ans. 2/27.]

9. A die is so loaded that the probability of a particular number appearing is proportional to the number. What is the probability of all single element events ? What is the probability of occurrence of an even number and of a number greater than 4 ?
[Ans. 1/21, 2/21, 3/21, 4/21, 5/21, 6/21, 12/21, 11/21.]

10. A coin is tossed until a head appears, or until it has been tossed 3 times. If the head does not appear on the first toss, find the probability that the coin is tossed 3 times.
[Ans. 1/2.]

11. Eight white and 2 black balls are randomly laid out in a row. What is the probability that two black balls are side by side ? What is the probability that they occupy the end positions ?
[Ans. 1/5, 1/45.]

12. An urn contains 10 white and 3 black balls white another contains 3 white and black balls. Two balls are transferred from the first urn to the second and then one ball is drawn from the latter, what is the probability that it is white ?
[Ans. 59/130.]

13. Find the value of *c* so that the following $f(x)$ is *p.d.f.*

$$f(x) = \begin{cases} c/x^2, & 10 \le x \le 20 \\ 0, & \text{otherwise.} \end{cases}$$

[Ans. $c = 20$.]

14. The following *p.d.f.* of the discrete random variable *x* represents the weekly demand of a certain item :

x	:	0	1	2	3
$P(x)$	:	0.15	0.25	0.35	0.45

If the weekly demands are independent and identical, find the *p.d.f.* for a two-week demand.

15. Eight coins were tossed together, and the number *x* of heads resulting was observed. The operation was performed 256 times, and the frequencies that were obtained for the different values of *x* are shown in the following table. Calculate measures of central tendency and mean deviation about mean–

x	:	0	1	2	3	4	5	6	7	8
f	:	1	9	26	59	72	52	29	7	1 (=256)

[Ans. $\bar{x} = 3{\cdot}973$, Mean deviation (about mean) = 1·089] **[IAS (Maths.) 99]**

14.18. MATHEMATICAL EXPECTATION OF A RANDOM VARIABLE

For a *discrete random variable X*, the expected value is denoted by $E(X)$ which is just the sum of the products of the possible values the random variable X takes on and their respective associated probabilities.

In other words, if the discrete random variable X takes n mutually exclusive values $x_1, x_2, \ldots, x_n$, and no others, with respective probabilities $p_1, p_2, p_3, \ldots, p_n$, the expected value of x is given by

$$E(x) = p_1x_1 + p_2x_2 + p_3x_3 + \ldots + p_nx_n = \sum_{i=1}^{n} p_i x_i . \qquad \ldots(14.24)$$

Similarly, for the continuous random variables, the expected value can also be obtained by the formula

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad \ldots(14.25)$$

where $f(x)$ is the probability density function.

For example, if x has an exponential distribution with parameter a, the expected value is given by

$$E(x) = \int_0^{\infty} x\,(f(x)\,dx. = \int_0^{\infty} x\,(ae^{-ax})\,dx = \frac{1}{a}\cdot$$

since the range for exponential distribution is 0 to ∞, and $f(x) = ae^{-ax}$.

Furthermore, we should also remember the following two expressions for the distribution function in *discrete* and *continuous* random variable cases :

(*i*) For a discrete random variable X, the *distribution* function denoted by $F(n)$, is given by

$$F(n) = P(X \le n) = \sum_{i=1}^{n} p_i. \qquad ...(14.26)$$

(*ii*) For a continuous random variable X, the distribution function, denoted for $F(a)$ is given by

$$F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)\,dx. \qquad ...(14.27)$$

Note. The mean of X is also called the **mathematical expectation** of X and is denoted by $E(X)$.

14.18-1. Illustrative Examples

Example 92. *Find the expected value of the number of points that will be obtained in a single throw with an ordinary die.*

Solution. Let X be the number of points obtained in a single throw with an ordinary die. Then x can take the values 1, 2, 3, 4, 5, 6.

Also, prob. of getting any number with a single throw = 1/6.

$\therefore$ Expected value of $x = 1 \times 1/6 + 2 \times 1/6 + 3 \times 1/6 + 4 \times 1/6 + 5 \times 1/6 + 6 \times 1/6$

$= 1/6\,(1 + 2 + 3 + 4 + 5 + 6) = 21/6 = 7/2$.

Example 93. *Thirteen cards are drawn simultaneously from a deck of 52. If aces count 1, face cards 10 and other according to denomination, find the expectation of the total score on the 13 cards.*

Solution. Let x_i be the number corresponding to the i^{th} card. Then x_i takes the values

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10

each having the prob. 1/13.

$$\therefore \quad E(x_i) = 1\cdot\frac{1}{13} + 2\cdot\frac{1}{13} + 3\cdot\frac{1}{13} + 4\cdot\frac{1}{13} + 5\cdot\frac{1}{13} + 6\cdot\frac{1}{13} + 7\cdot\frac{1}{13} + 8\cdot\frac{1}{13} + 9\cdot\frac{1}{13} + 10\cdot\frac{1}{13} + 10\cdot\frac{1}{13} + 10\cdot\frac{1}{13} + 10\cdot\frac{1}{13}$$

$$= 1/13\,(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 10 + 10 + 10) = 85/13.$$

Example 94. *A and B play with one die for a price of Rs. 11 which is to be won by a player who first throws 6. If A has the first throw, what are their respective expectations ?*

Solution. A can win at the 1st, 3rd, 5th, ..., mutually exclusive trials with respective probabilities

$$\frac{1}{6},\ \left(\frac{5}{6}\right)^2\frac{1}{6},\ \left(\frac{5}{6}\right)^4\frac{1}{6},\ \ldots\ldots$$

$$P(A) = \frac{1}{6} + \left(\frac{5}{6}\right)^2\frac{1}{6} + \left(\frac{5}{6}\right)^4\frac{1}{6} + \ldots\ldots = \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \ldots\ldots\right]$$

$$= \frac{1}{6}\times\frac{1}{1 - 25/36} = \frac{1}{6}\times\frac{36}{11} = \frac{6}{11}\cdot \left(S = \frac{a}{1-r} \text{ in a G.P. with first term } a \text{ and common ratio } r\right)$$

$$\therefore \quad P(B) = 1 - 6/11 = 5/11\cdot$$

Therefore, A's expectation = $6/11 \times 11$ = Rs. 6, and B's expectation = $5/11 \times 11$ = Rs. 5.

Example 95. *A doctor recommends a patient to go on a particular diet for two weeks and there is equal likelihood for the patient to lose his weight between 2 kgs. and 4 kgs. What is average amount the patient is expected to lose on this diet.*

• **Solution.** Here $f(x) = 1/2$, $2 < x < 4$ and $f(x) = 0$, otherwise.

The weight expected to be lost, $E(x) = \int_2^4 x\,.\,1/2\,dx = \left[x^2/4\right]_2^4 = 1/4\,[4^2 - 2^2] = 3$ kg.

Example 96. *Two persons A and B, play the game of tossing a die with faces marked 1 to 6. He who first gets face 1, wins the game. If A begins the game and each player wins an amount of money equal to the amount of tosses required to win, find their respective mathematical expectations.*

Solution. The game can be won by A, (*i*) if he gets face 1 in the first throw;
(*ii*)if A and B do not get face 1 in the first throw and A gets it in the second throw;
(*iii*)if A and B do not get face 1 in the first and second throws and A gets it in the third throw; and so on.
Now probability of A winning in the first throw = $1/6$.

Probability of A winning in the second throw = $5/6 \cdot 5/6 \cdot 1/6 = (5/6)^2 \cdot 1/6$.

Probability of A winning in the third throw = $5/6 \cdot 5/6 \cdot 5/6 \cdot 5/6 \cdot 1/6 = (5/6)^4 \cdot 1/6$.

Probability of A winning in the fourth throw = $(5/6)^6 \cdot 1/6$, and so on.

$\therefore$ A's mathematical expectation will be given by

$$E(x) = 1 \cdot 1/6 + 3 \cdot (5/6)^2 \cdot 1/6 + 5 \cdot (5/6)^4 \cdot 1/6 + 7 \cdot (5/6)^6 \cdot 1/6 + \dots \dots \qquad \dots(i)$$

This is an arithmetico-geometric progression with $(5/6)^2$ as the common ratio for the geometric progression.

$$(5/6)^2 E(x) = (5/6)^2 \cdot 1/6 + 3 \cdot (5/6)^4 \cdot 1/6 + 5 \cdot (5/6)^6 \cdot 1/6 + \dots \dots \qquad \dots(ii)$$

Subtracting the eqn. (*ii*) from (*i*), we get

$$11/36\, E(x) = 1 \cdot 1/6 + 2 \cdot (5/6)^2 \cdot 1/6 + 2 \cdot (5/6)^4 \cdot 1/6 + 2 \cdot (5/6)^6 \cdot 1/6 + \dots \dots$$
$$= 1/6 + 2 \cdot (5/6)^2 \cdot 1/6\, [\cdot 1 + (5/6)^2 + (5/6)^4 + \dots \dots]$$
$$= \frac{1}{6} + \frac{50}{216}\left[\frac{1}{1-(5/6)^2}\right] = \frac{1}{6} + \frac{50}{216} \times \frac{36}{11} = \frac{61}{66}.$$
$$\therefore \quad E(x) = \frac{61}{66} \times \frac{36}{11} = \frac{366}{121}.$$

Similarly, B's mathematical expectation is given by

$$E(y) = 2 \cdot (5/6) \cdot 1/6 + 4 \cdot (5/6)^3 \cdot 1/6 + 6 \cdot (5/6)^5 \cdot 1/6 + \dots \dots$$
$$\therefore \quad (5/6)^2 E(y) = 2 \cdot (5/6)^3 \cdot 1/6 + 4 \cdot (5/6)^5 \cdot 1/6 + 6 \cdot (5/6)^7 \cdot 1/6 + \dots \dots$$

Subtracting, we get

$$11/36\, E(y) = 2 \cdot (5/6) \cdot 1/6 + 2 \cdot (5/6)^3 \cdot 1/6 + 2 \cdot (5/6)^5 \cdot 1/6 + \dots \dots$$
$$= 2 \cdot (5/6) \cdot 1/6\, [\cdot 1 + (5/6)^2 + (5/6)^4 + \dots \dots]$$
$$= \frac{10}{36}\left[\frac{1}{1-(5/6)^2}\right] = \frac{10}{36} \times \frac{36}{11} = \frac{10}{11}.$$
$$\therefore \quad E(y) = \frac{10}{11} \times \frac{36}{11} = \frac{360}{121}.$$

14.19. CENTRAL TENDENCY

After collecting the data for a random variable x and analysing it in the form of frequency distribution, the next necessary step is to find the nature of distribution. In this context, a property for values of x to tend towards the centre is quite important. This property is called the ***Central Tendency.*** There are three most important measures of central tendency : (*i*) mean, (*ii*) mode, (*iii*) median.

Mean. *If x is the random variable, then its expected value E(x) itself is called the* ***mean*** *or average value of x and denoted by* $\bar{x}$. *Mean value of the random variable locates the middle of its probability function.*

Mode. *The mode of a random variable x is that value of the variable which occurs with the greatest frequency and is denoted by* $\hat{x}$. *It is possible that a particular distribution may not have a mode, or if it has a mode, it may not be unique.*

A distribution is called unimodal, bimodal, trimodal, ... depending upon whether it has one, two, three .. ... modes.

For a discrete distribution, mode $\hat{x}$ is determined by the following inequalities :
(i) $P(x = x_i) \le (x = \hat{x}), x_i \le \hat{x}$, (ii) $P(x = x_j) \le (x = \hat{x}), x_j \le \hat{x}$.

For a continuous distribution it is determined by the following equations/inequalities :

$$\frac{d}{dx}[f(x)] = 0 \text{ and } \frac{d^2}{dx^2}[f(x)] = 0.$$

Median. *For a discrete or continuous distribution of a random variable x, the median is defined as the variate-value X satisfying the inequations :* $P(x \le X) = ½$ and $P(x \ge X) = ½$.

It is denoted by $\tilde{x}$.

If a continuous distribution function has a *p.d.f.* $f(x)$ in the range (a, b) then $\tilde{x}$ is given by

$$\int_a^x f(x)\,dx = ½ = \int_b^x f(x)\,dx.$$

Remark. If mean and median are known, then the mode can be calculated from the empirical formula

Mean – Mode = 3(Mean – Median).

SELF EXAMINATION PROBLEMS

1. What is the chance of throwing a total of 3 or 5 or 11 with two dice ?
2. *A, B, C* in order cut a pack of cards, replacing them after each cut. Find their respective chances of first cutting a heart.
3. Three cards are drawn from an ordinary pack. Find the chance that they consists of a knave, a queen and a king.
4. The odds against *A* solving a certain problem are 4 to 3 and odds in favour of *B* solving the same are 7 to 5. What is the chance that the problem will be solved if they both try ?
5. Four cards are drawn without replacement. What is the probability that (*a*) they are all aces ? (*b*) they are all of different suits ?
6. In shuffling a pack of cards, three are accidentally dropped. Find the chance that the missing cards should be from different suits.
7. A coin is tossed three times. Find the probability of getting head and tail alternately.
8. A bag contains 3 white, 4 red and 6 green balls. If only one ball is to be drawn, what is the chance of its being neither a red ball nor a green ball ?
9. An integer is chosen at random from 1 to 100. What is the probability that
   (*i*) it is divisible by 3, (*ii*) it is divisible by 8, (*iii*) it is divisible by 3 or 8 ?
10. Discuss and criticise the following :
    $$P(A) = 2/3,\ P(B) = 1/4,\ P(C) = 1/6$$
    where *A*, *B* and *C* are mutually exclusive events.
11. Show that in a single throw with two dice the chances of throwing more than 7 is equal to that of throwing less than 7 each being 5/12.
12. A speaks truth in 75% and *B* in 80% of the cases. In what percentages of cases are they likely to contradict each other in stating the same fact ?
    [**Ans.** 7/20]
13. What is the probability of obtaining 2 heads and two tails when 4 coins are thrown ?
    [**Ans.**3/8]
14. If the chance that a vessel arrives at a port is 9/10, find the chance that out of 5 vessels expected, 4 at least will arrive safely ?
    [**Ans.** 0·91854]
15. If on an average 9 ships out of 10 return safe to port, what is the chance that out of 5 ships expected at least 3 will arrive ?
    [**Ans.** 0·99144]
16. In four throws, with a pair of dice what is the chance of throwing doublets twice at least ?
    [**Ans.** 19/144]
17. In tossing 10 coins, what is the probability of having exactly 5 heads ?
    [**Ans.** 63/256]
18. Twelve per cent of a given lot of manufactured goods are defective. What is the probability that in a sample of five such goods exactly one will be defective?
    [**Ans.** 0·3598]
19. Three per cent of a given lot of manufactured parts are defective. What is the probability that in a sample of four items non will be defective ?
    [**Ans.** 0·8853]
20. A dice is thrown 120 times and getting '1' or '5' is considered a success. Find the mean and the variance of the number of successes.
    [**Ans.** Mean = 40, Variance = 80/3]
21. Four coins are tossed. Find the mean and the variance of the number of heads obtained.
    [**Ans.** Mean = 2, Variance = 1]
22. Two cards are drawn simultaneously (without replacement) from a well-shuffled pack of 52 cards. Find μ and σ for the number of aces.
    [**Ans.** Mean = 34/221, Variance = 400/4873]

23. (*a*) If ten fair coins are tossed, what is the probability that there are (*i*) exactly 3 heads, (*ii*) not more than 3 heads.
(*b*) The probability that an evening college student will be graduate is 0.4. Determine the prob. that out of 5 students (*i*) none, (*ii*) one and (*iii*) atleast one will be graduate.
[**Ans.** (a) (i) 15/128, (ii) 11/64; (b) (i) 0·07776, (iv) 0·2592, (iii) 0·9224]

24. Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 20% of such fuses are defective.
[**Ans.** 11817433 e^{-40}]

25. A perfect cubic die is thrown a large number of times in sets of 8. The occurrence of 5 or 6 is called a success. In what proportion of the sets would you expect 3 success ?
[**Ans.** 1792/6561]

26. In a certain factory turning out razor blades there is a small chance of 1/100 for any blade to be defective. The blades are supplied in packets of ten. In a consignment of 10,000 packets from the factory, how many packets are expected to have one defective blade ?
[**Ans.** 0·09135]

27. A square sheet of tin, 20 centimeters wide, that contains 10 rows and 10 columns of circular holes, each 1 centimeter in diameter, with centres evenly spaced at a distance 2 centimeters apart. What is the probability :
(*i*) that a particle of sand (considered as a point) blown against the tin sheet will fall upon one of the holes and thus pass through;
(*ii*) that a ball of diameter 0.5 cm. thrown upon the sheet will pass thought without hitting the tin sheet.
[**Ans.** (i) 11/56, (ii) 11/224] **[I.A.S. (Maths.) 81]**

28. A bag contains 10 balls, either black or white, but it is not known how many of each. A ball is drawn at random and is white. If a second ball is drawn at random (without the first ball being returned to the bag), what is the probability, it also will be white ?
[**Ans.** (a – 1)/9, where a denotes the no. of white balls.] **[I.A.S. (Maths.) 82]**

29. A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Determine the probability that it is (*i*) red, (*ii*) white, (*iii*) blue, (*iv*) not red, (*v*) red or white.
[**Ans.** (i) 6/15, (ii) 4/15, (iii) 5/15, (iv) 9/15, (v) 10/15] **[I.A.S. (Maths.) 84]**

30. Two cards are drawn successively from a pack without replacing the first. If the first card is a spade, find the probability that the second card is also a spade. Find also the probability that both cards are spades. **[I.A.S. (Maths.) 85]**
[**Ans.** 1/17]

31. A and B throw alternately with a pair of ordinary dice. A wins if he throws 6 before B throws 7, and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61.
[**Ans.** 1/17] **[I.A.S. (Maths.) 87]**

32. If n letters are randomly placed in correctly addressed envelopes, prove that the probability that exactly r letters are placed in correct envelopes is given by

$$\frac{1}{r!}\sum_{k=0}^{n-r}(-1)^k\frac{1}{k!}\,;\ r=1,2,..,n.$$ **[I.A.S. (Maths.) 89]**

33. Show that the expectation of the sum of two stochastic variates is equal to the sum of their expectations.
[I.A.S. (Maths.) 88]

34. A box contains a white and b balck balls. c ball are drawn. Find the expectation of the number of white balls drawn.
[I.A.S. (Maths.) 88]

35. The contents of urns A, B, C are as follows :
Urn A : 1 white, 2 black and 3 red balls.
Urn B : 2 white, 1 black and 1 red balls.
Urn C : 4 white, 5 balck and 3 red balls.
One urn is chosen at random and two balls are drawn. They happen to be white and red. What are the probabilities of their having come from Urns A, B and C ? What is the probability that they came from urn B or C ?
[**Ans.** 33/118, 55/118, 30/118, 85/118] **[I.A.S. (Maths.) 94, 89]**

36. A bag contains a coin of value M and a number of other coins whose total value is m. A person draws one at a time till he draws the coin M. Find the value of his expectation.
[**Ans.** $(M+\frac{1}{2}m)$] **[I.A.S. (Maths.) 89]**

37. An urn contains 3 white, 5 black and 2 red balls. Two persons draw the ball tern by tern without replacement. The person who will draw the white ball first will be considered the winner. The game will be considered without decision if red ball is drawn.
Assuming that : A_1 = {the person who starts the game is the winner}, A_2 = {the second player is the winner}, and B = {the game is without decision}; find $P(A_1)$, $P(A_2)$ and $P(B)$.
[**Ans.** (10/17, 7/17, 2/17)] **[I.A.S. (Maths). 90]**

38. Balls are taken one by one out of an urn containing a white balls and b black balls. What is the expectation of the number of black balls preceding the first white ball ? **[I.A.S. (Maths.) 90]**

39. A box contains k varieties of objects, the number of objects of each variety being the same. Then objects are drawn one at a time and put back before the next drawn. Denoting by n the smallest number of drawings which produce objects of all varieties, find $E(n)$, the expectation of n. **[I.A.S. (Maths.) 90]**

40. Find the expectation of the number of failures preceding the first success in an infinite series of independent trials, with constant probability p of success in each trial.
[Ans. $(1-p)/p$] **[IAS (Maths) 91]**

41. A building contractor receives bricks from 3 different suppliers — 35% from supplier A, 45% from supplier B and the remaining from supplier C. 90% of bricks supplied by A, 80% of those supplied by B and 95% of those supplied by C are according to specifications. A brick drawn at random is not according to specification. What is the probability that it came from B ?
[Ans. 2/3] **[IAS (Maths) 91]**

42. Each coefficient in the equation $Ax^2 + Bx + C = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots.
[Ans. 43/216] **[IAS (Maths) 92]**

43. Prove that if A, B and C are random events in a sample space and if A, B, C are pairwise independent and A is independent of $B \cup C$, then A, B and C are mutually independent. **[IAS (Maths) 92]**

44. An event A is known to be independent of the events B, $B \cup C$ and $B \cap C$. Show that it is also independent of C. **[IAS (Maths) 93]**

45. If the probability that an individual suffers a bad reaction from injection of a given serum is 0·001, determine the probability that out of 2000 individuals : (i) exactly 3 will suffer a bad reaction (ii) more than 2 will suffer a bad reaction.
[Ans. (i) $4e^{-2}/3$, (ii) $1 - 4e^{-2}$] **[IAS (Maths) 93]**

46. A and B play a game of dice. A wins if he throws 11 with 3 dice before B throws 7 with 2 dice. B wins if he throws 7 before A throws 11. A starts the game and they throw alternately. What are the odds against A winning the game ultimately ?
[Ans. 7 : 6] **[IAS (Maths.) 95]**

47. If x be one of the first hundred netural numbers chosen at random, find the probability that

$$x + \frac{100}{x} > 50.$$

[IAS (Maths.) 96]

48. A printing machine can print n 'letters', say $\alpha_1, \alpha_2, \ldots, \alpha_n$. It is operated by electrical impulses, each letter being produced by a different impulse. Assume that there exists a constant probability p of printing the correct letter and also asuume independence. One of the n impulses chosen at random, was fed into the machine twice and both times the letter α_1 was printed. compute the probability that the impulse chosen was meant to print α_1. **[IAS (Maths.) 99]**

49. If two independent variates, x and y have Poisson distributions with means m_1 and m_2, find the distribution of the sum $x + y$.
[Ans. $x + y \sim P(m_1 + m_2)$] **[IAS (Maths.) 99]**

50. Two unbiased coins are tossed once (independently) and the number x of heads that turned up is noted. A number is selected at random from x, $x + 1$ and $x + 2$. It Y is the number selected, find the joint distribution of x and y. Also obtain the expectation of xy. **[IAS (Maths.) 2000]**

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15

PROBABILITY DISTRIBUTIONS

15.1. INTRODUCTION

Any rule which assigns probabilities to each of the possible values of a random variable is called a ***probability distribution.*** There are several different ways to specify probability distribution.

For discrete random variables, the most obvious and commonly used method of specifying the rule is to indicate the probability for each value separately. The function $P(x)$, defined as $P(x) = P(X = x)$, is called the *probability distribution function.*

For continuous random variables, the situation is some what complicated because the range of possible values is uncountably infinite. In this case, the distribution is defined by the ***probability density function*** $f(x)$ for the prescribed range of random variable X.

The ***probability density function,*** $f(x)$, is a function which, when integrated between a and b, gives the probability that the random variable will assume a value between a and b. That is,

$$P(a \le X \le b) = \int_a^b f(x)\, dx.$$

In this chapter, we shall describe some important probability distributions (***discrete*** and ***continuous***) which are commonly used in various fields of *Operations Research.*

I–Discrete Probability Distributions

15.2. THE DISCRETE UNIFORM DISTRIBUTION

If a random variable X has only a finite number of possible values, each of which can occur with equal likelihood, then such distribution is said to ***discrete uniform.*** With no serious loss to generality, it can be assumed that the range of X is $x = 1, 2, ..., N$, in which case the probability distribution function is given by

$$P(x) = \frac{1}{N}, x = 1, 2, ..., N$$

when X has this range, the mean and variance are given by the formulae

$$E(x) = \frac{N+1}{2}, \quad V(x) = \frac{N^2 - 1}{12}.$$

In any case, the probability density function is just one divided by the total number of possible values, for each value, and the expectation falls at the midpoint of the range. Although this distribution has various applications, the discrete uniform distribution is not so important as it is frequently thought to be by beginners in probability. It is important to note that the discrete uniform distribution is just one of many useful distributions.

15.3. THE BERNOULLI DISTRIBUTION

When a random variable must assume one of two values, 0 or 1, such variable is called a ***Bernoulli random variable.*** The corresponding experiment, which has only two possible outcomes, is said to be a ***Bernoulli trial.*** Usually the outcome which is mapped by the random variable into the value '1' is named as a 'success' the other is called a 'failure'. The probability distribution is given by $P(1) = p$, $P(0) = 1 - p$. where p is the only parameter of the distribution, usually referred to as the *'probability of success'.*

This distribution may appear so trivial requiring no special attention. Although the direct applications of this distribution are very limited, but a number of more important distributions can be obtained by considering a sequence of independent *Bernoulli Trials*.

15.4. THE BINOMIAL DISTRIBUTION

This distribution is due to *James Bernoulli (published in 1713, eight years after his death)*.

Bernoulli trials. A series of independent trials which can result in one of the two mutually exclusive possibilities–success or failure–such that the probability of success (or failure) in each trial is constant, then such repeated independent trials are called as *Bernoulli trials.*

If we perform a series of n *Bernoullian trials* such that for each trial, p is the prob. of success and q is the prob. of failure ($p + q = 1$), then probability of r successes is a series of n independent trials

$$P(r) = {}^nC_r p^r q^{n-r}$$

where $p + q = 1$, and $r = 0, 1, 2, 3, \ldots, n$.

This is known as *binomial distribution.* The constant n and p (or q) which appear in the binomial distribution are called the *parameters.*

$$P(0) = q^n, \quad P(1) = {}^nC_1 p\, q^{n-1}, \quad P(2) = {}^nC_2 p^2 q^{n-2}, \ \ldots \ \ldots \qquad \ldots, \ P(n) = p^n.$$

The probabilities of 0 success, 1 success, 2 successes, ..., n success are nothing but the first, the second, the third, ..., the $(n + 1)$th terms in the binomial expansion $(q + p)^n$. For this reason, the distribution is called *Binomial Distribution* (B.D.).

Mean and Variance of Binomial Distribution

$D = n - 0$	f	fD	fD^2
0	q^n	0	0
1	${}^nC_1 pq^{n-1}$	npq^{n-1}	npq^{n-1}
2	${}^nC_2 p^2 q^{n-2}$	$n(n-1)p^2q^{n-2}$	$2n(n-1)p^2q^{n-2}$
3	${}^nC_3 p^3 q^{n-3}$	$\frac{n(n-1)(n-2)}{2.1} p^3 q^{n-3}$	$\frac{3n(n-1)(n-2)}{2.1} p^3 q^{n-3}$
...	... ...	... ...	... ...
...	... ...	... ...	... ...
n	p^n	np^n	n^2p^n
Total	1	np	$np[1 + p(n-1)]$

$$\Sigma f = q^n + {}^nC_1 q^{n-1} p + {}^nC_2 q^{n-2} p^2 + {}^nC_3 q^{n-3} p^3 + \ldots + p^n = (q + p)^n = (1)^n = 1 \quad (\because \ q + p = 1)$$

$$\Sigma fD = 0 + nq^{n-1}p + n(n-1)\, q^{n-2}p^2 + \frac{n(n-1)(n-2)}{2\cdot 1} q^{n-3}p^3 + \ldots + np^n$$

$$= np\left[q^{n-1} + (n-1)\, q^{n-2}p + \frac{(n-1)(n-2)}{2\cdot 1} q^{n-3}p^2 + \ldots + p^{n-1}\right]$$

$$= np(q+p)^{n-1} = np(1)^{n-1} = np. \qquad (\because \ q + p = 1)$$

Thus, ***Mean of B.D.*** $= a + \frac{\Sigma fD}{\Sigma f} = 0 + \frac{np}{1} = \boldsymbol{np}$, where '$a$' is the assumed mean.

$$\text{Now, } \Sigma fD^2 = 0 + nq^{n-1}p + 2n(n-1)q^{n-2}p^2 + \frac{3n(n-1)(n-2)}{2\cdot 1} q^{n-3}p^3 + \ldots + n^2p^n$$

$$= np\left[q^{n-1} + 2(n-1)q^{n-2}p + \frac{3(n-1)(n-2)}{2\cdot 1} q^{n-3}p^2 + \ldots + np^{n-1}\right]$$

$$= np\left[\left\{ q^{n-1} + (n-1)\, q^{n-2}p + \frac{(n-1)(n-2)}{2\cdot 1} q^{n-3}p^2 + \ldots + p^{n-1}\right\}\right.$$

$$\left. + \left\{ (n-1)\, q^{n-2}p + \frac{2(n-1)(n-2)}{2\cdot 1} q^{n-3}p^2 + \ldots + (n-1)\, p^{n-1}\right\}\right]$$

$$= np\,[(q+p)^{n-1} + (n-1)p\,\{q^{n-2} + (n-2)q^{n-3}p + \ldots + p^{n-2}\}]$$
$$= np\,[1 + (n-1)\,p\,(q+p)^{n-2}] = np\,[1 + (n-1)\,p(1)^{n-2}] = np\,[1 + (n-1)p].$$

Now, ***Variance*** $= \dfrac{\Sigma fD^2}{\Sigma f} - \left(\dfrac{\Sigma fD}{\Sigma f}\right)^2 = \dfrac{np[1+(n-1)p]}{1} - \left(\dfrac{np}{1}\right)^2$

$$= np[1 + np - p] - n^2p^2 = np + n^2p^2 - np^2 - n^2p^2 = np - np^2 = np(1-p) = npq \qquad (\because p + q = 1)$$

$\therefore$ ***Variance of B.D. = npq,*** and hence ***S.D.*** $= +\sqrt{npq}$.

15.4-1. Recurrence Formula for Binomial Distribution

We have $\quad P(r) = {}^nC_r\,p^r q^{n-r} \quad$ and $\quad P(r+1) = {}^nC_{r+1}\,p^{r+1}\,q^{n-r-1}$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}\,p^{r+1}q^{n-r-1}}{{}^nC_r\,p^r q^{n-r}} = \frac{n\,!}{(r+1)\,!\,(n-r-1)\,!} \times \frac{r\,!\,(n-r)\,!}{n\,!} \cdot p^{(r+1)-r} \cdot q^{(n-r-1)-(n-r)}$$
$$= \frac{n-r}{r+1} \cdot \frac{p}{q}$$
$$\therefore \quad \boldsymbol{P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q}\,P(r)} \cdot$$

If $P(0)$ is known, we can easily find $P(1)$, $P(2)$......., etc. by applying above recurrence formula.

Q. Define Binomial Distribution. Calculate mean and variance of Binomial Distribution.

15.4-2. Illustrative Examples

Example 1. *Six dice are thrown 729 times. How many times do you expect at least three dice show a 5 or 6 ?*

Solution. Probability of appearing 5 or 6 on any dice is given by $p = 2/6 = 1/3$.

Since there are six trials in each set, the number of successes vary from 1 to 6 and thus follow the binomial distribution with $x = 6$ and $p = 1/3$.

Hence the probability of at least three successes $= 729 \times 233/729 = 233$ ($\because$ total no. of throws $= 729$) **Ans.**

Example 2. *Determine the B.D. whose mean is 9 and whose standard deviation is* $3/2$.

Solution. *mean* $= np = 9$, *variance* $= (S.D.)^2 = npq = (3/2)^2$.

Therefore, $\quad \dfrac{npq}{np} = \dfrac{9/4}{9} = \dfrac{1}{4}$ or $q = 1/4$ and hence $p = 1 - q = 1 - 1/4 = 3/4$.

Now since $np = 9$, $\therefore n \times 3/4 = 9$ or $n = \dfrac{9 \times 4}{3} = 12$. Therefore, $n = 12, p = 3/4, q = 1/4$.

Example 3. *If ten fair coins are tossed, what is the probability that there are (a) exactly 3 heads (b) not more than 3 heads ?*

Solution. The probability of head with a coin is $1/2$. Probability that of 10 coins, r coins show heads is given by

$$P(r) = {}^{10}C_r\,(1/2)^r\,(1/2)^{10-r}, \qquad r = 0, 1, 2, \ldots, 10.$$

(*a*) For $r = 3$, $\quad P(3) = {}^{10}C_3\,(1/2)^3\,(1/2)^{10-3} \quad = {}^{10}C_3\left(\dfrac{1}{2}\right)^{10} = 120/1024 = 15/128.$

(*b*) Probability of not more than 3 heads is given by

$$P(0) + P(1) + P(2) + P(3) = (1/2)^{10}\,({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3) = 176/1024 = 11/64.$$

Example 4. *If the sum of the mean and the variance of a binomial distribution for 5 trials is 1.8, find the distribution.*

Solution. *Mean* $= np$ and *Variance* $= npq$.

$\therefore \quad np + npq = 1.8 \quad$ or $\quad np(1+q) = 1.8 \quad$ or $\quad 5p[1 + (1-p)] = 1.8 \quad$ or $\quad 5p(2-p) = 1.8$

or $\quad 10p - 5p^2 = 1.8 \quad$ or $\quad 5p^2 - 10p + 1{\cdot}8 = 0 \quad$ or $\quad p^2 - 2p + 0.36 = 0$. Solving, we get

$$p = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 0.36}}{2 \times 1} = \frac{2 \pm \sqrt{4 - 1.44}}{2} = \frac{2 \pm \sqrt{2.56}}{2} = \frac{2 \pm 1.6}{2}$$
$$= 1 \pm 0.8 = 1.8 \text{ or } 0.2.$$

But, p cannot be equal to 1.8. Therefore, $p = 0.2 = 1/5$, $q = 1 - 1/5 = 4/5$. Hence $n = 5$, $p = 1/5$ and $q = 4/5$.

Example 5. *The probability of a man hitting a target is 1/4. If he fires seven times, what is the probability of his hitting the target at least twice ?*

Solution. Here $p = 1/4$, $n = 7$, $q = 1 - 1/4 = 3/4$,. and $P(r) = {}^7C_r \, (1/4)^r \, (3/4)^{7-r}$.

$\therefore$ Probability of hitting the target at least twice

$$= P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1 - [P(0) + P(1)]$$
$$= 1 - [\,{}^7C_0 \, (1/4)^0 \, (3/4)^7 + {}^7C_1 \, (1/4)^1 \, (3/4)^6]$$
$$= 1 - [\,1 \times 1 \times (3/4)^7 + 7 \times 1/4 \times (3/4)^6]$$
$$= 1 - (3/4)^6 \, [3/4 + 7/4] = 1 - (3/4)^6 \times 10/4 \,. \qquad \textbf{Ans.}$$

Example 6. *If the chance that any of the 5 telephone lines is busy at an instant is 0.01, what is the prob. that all the lines are busy ? What is the prob. that not more than 3 lines are busy ?*

Solution. Here $p = 0.01 = 1/100$, $q = 1 - 1/100 = 99/100$.

P (all lines are busy) $= P(5) = {}^5C_5 \, (1/100)^5 \, (99/100)^{5-5} = (1/100)^5$.

P (not more than 3 lines are busy) $= 1 - [P(4) + P(5)] = 1 - [\,{}^5C_4 \, (1/100)^4 \, (99/100)^{5-4} + {}^5C_5 \, (1/100)^5 \, (99/100)^0]$

$$= 1 - [5 \times (1/100)^4 \, (99/100) + 1 \times (1/100)^5] = 1 - [\,5 \times 99 \times (1/100)^5 + (1/100)^5]$$
$$= 1 - (495 + 1) \, (1/100)^5 = 1 - (496) \, (1/100)^5.$$

Example 7. *A bag contains 10 balls each marked with one of the digits 0 to 9. If 4 balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0 ?*

Solution. Here $p = 1/10$, $q = 1 - 1/10 = 9/10$, $n = 4$. 'Prob. that none is marked' is given by

$$P(0) = {}^4C_0 \, (1/10)^0 \, (9/10)^{4-0} = 1 \times 1 \times (9/10)^4 = (9/10)^4.$$

Example 8. *A die is thrown 3 times. If getting a 'six' is considered a success, find the probability of (i) 3 successes, (ii) at least two successes.*

Solution. Here $p = 1/6$, $q = 1 - 1/6 = 5/6$, $n = 3$. We have,

(*i*) $P(3) = {}^3C_3 \, (1/6)^3 \, (5/6)^{3-3} = 1 \times (1/6)^3 \times 1 = (1/6)^3 = 1/216$.

(*ii*) P (at least two successes) $= P(2) + P(3) = {}^3C_2 \, (1/6)^2 \, (5/6)^{3-2} + {}^3C_3 \, (1/6)^3 \, (5/6)^{3-3}$

$$= 3 \times 5/216 + 1/216 = 16/216 = 2/27.$$

Example 9. *A product is supposed to contain 5% defective items. What is the probability that a sample of 8 items will contain less than 2 defective items.*

Solution. Here $p = 5/100 = 1/20$, $q = 1 - p = 1 - 1/20 = 19/20$, $n = 8$.

$\therefore$ Required probability $= P(0) + P(1) = {}^8C_0 \, (1/20)^0 \, (19/20)^{8-0} + {}^8C_1 \, (1/20)^1 \, (19/20)^{8-1}$

$$= 1 \times 1 \times (19/20)^8 + 8 \times (1/20) \, (19/20)^7 = (19/20)^7 \, [19/20 + 8/20] = (19/20)^7 \, (27/20).$$

Example 10. *If on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive, 4 at least will arrive safely.*

Solution. Let p be the probability of a vessel being wrecked. Here $p = 1/10$, $q = 1 - 1/10 = 9/10$, $n = 5$.

$\therefore$ $P(r) = {}^5C_r \, (1/10)^r \, (9/10)^{5-r}$, $r = 0, 1, 2, 3, 4, 5$.

P (at least 4 vessels will arrive safely) $= P(0) + P(1)$

$$= {}^5C_0 \, (1/10)^0 \, (9/10)^5 + {}^5C_1 \, (1/10)^1 \, (9/10)^{5-1} = 1 \times 1 \times (9/10)^5 + 5 \times (1/10) \, (9/10)^4$$
$$= \frac{1}{(10)^5} (9^5 + 5 \times 9^4) = \frac{9^4 \times 14}{(10)^5} = 0.91854. \qquad \textbf{Ans.}$$

Example 11. *The probability that a bulb produced by a factory will fuse after 100 days of use is 0.05. Find the probability that out of 5 such bulbs, (i) none (ii) not more than one (iii) more than one (iv) at least one will fuse after 100 days of use.*

Solution. $p = 0.05 = 5/100 = 1/20$, $q = 1 - p = 1 - 1/20 = 19/20$, $n = 5$.

(*i*) $P\,(\text{none}) = P(0) = {}^5C_0\,(1/20)^0\,(19/20)^{5-0} = 1 \times 1 \times (19/20)^5 = (19/20)^5$.

(*ii*) $P\,(\text{not more than one}) = P(0) + P(1) = {}^5C_0\,(1/20)^0\,(19/20)^5 + {}^5C_1\,(1/20)^1\,(19/20)^4$

$$= (19/20)^5 + 5\,(1/20)\,(19/20)^4 = (19/20)^4\,[19/20 + 5/20] = (24/20) \times (19/20)^4 = (6/5) \times (19/20)^4.$$

(*iii*) $P\,(\text{more than one}) = 1 - P\,(\text{not more than one}) = 1 - (6/5) \times (19/20)^4$

(*iv*) $P\,(\text{at least one}) = 1 - P(0) = 1 - (19/20)^5$.

Example 12. *Calculate $P(r)$ for $r = 1, 2, 3, 4$ and 5, taking $n = 5$ and $p = 1/6$ with the help of the recurrence formula of the B.D.*

Solution. $P(r+1) = \dfrac{n-r}{r+1} \cdot \dfrac{p}{q}\,P(r)$. Given $n = 5, p = 1/6$, $\therefore\ q = 1 - 1/6 = 5/6$.

$$\therefore\quad P(r+1) = \frac{5-r}{r+1} \cdot \frac{1/6}{5/6}\,P(r) = \frac{5-r}{r+1} \times \frac{1}{5} \times P(r)$$

Putting $r = 0, 1, 2, 3, 4,$ we get

$$P(1) = \frac{5-0}{0+1} \times \frac{1}{5} \times P(0) = P(0) = q^n = \left(\frac{5}{6}\right)^5 = \frac{3125}{7776} = 0.4018.$$

$$P(2) = \frac{5-1}{1+1} \times \frac{1}{5} \times P(1) = \frac{4}{2} \times \frac{1}{5} \times \frac{3125}{7776} = \frac{1250}{7776} = 0.1007.$$

$$P(3) = \frac{5-2}{2+1} \times \frac{1}{5} \times P(2) = \frac{3}{3} \times \frac{1}{5} \times \frac{1250}{7776} = \frac{250}{7776} = 0.0321.$$

$$P(4) = \frac{5-4}{3+1} \times \frac{1}{5} \times P(3) = \frac{2}{4} \times \frac{1}{5} \times \frac{250}{7776} = \frac{25}{7776} = 0.0032.$$

$$P(5) = \frac{5-4}{4+1} \times \frac{1}{5} \times P(4) = \frac{1}{5} \times \frac{1}{5} \times \frac{25}{7776} = \frac{1}{7776} = 0.00013.$$

Example 13. *In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is 5/6. What is the probability that he will knock down fewer than 2 hurdles ?*

Solution. Given $n = 10$, prob. that he will clear each hurdle, *i.e.* $q = 5/6$, and prob. that he will knock down, $p = 1 - 5/6 = 1/6$.

$\therefore\quad P\,(\text{knocking down fewer than 2 hurdles}) = P(0) + P(1) = {}^{10}C_0\,(1/6)^0\,(5/6)^{10} + {}^{10}C_1\,(1/6)^1\,(5/6)^9$

$$= 1 \times 1 \times (5/6)^{10} + 10\,(1/6)\,(5/6)^9 = (5/6)^9\,[5/6 + 10/6] = (15/6)\,(5/6)^9.$$

Example 14. *Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that (i) all the five cards are spades ? (ii) only 3 cards are spades ? (iii) none is a spade ?*

Solution. Here $p = 13/52 = 1/4$, $q = 1 - 1/4 = 3/4$, $n = 5$.

(*i*) $P(5) = {}^5C_5\,(1/4)^5\,(3/4)^{5-5} = 1 \times (1/4)^5 \times 1 = (1/4)^5$.

(*ii*) $P(3) = {}^5C_3\,(1/4)^3\,(3/4)^{5-3} = 10 \times (1/4)^3\,(3/4)^2 = \dfrac{10 \times 9}{(4)^5} = \dfrac{90}{(4)^5}$.

(*iii*) $P(0) = {}^5C_0\,(1/4)^0\,(3/4)^{5-0} = 1 \times 1 \times (3/4)^5 = (3/4)^5$.

15.5. THE POISSON DISTRIBUTION

Poisson distribution is due to ***S.D. Poisson*** *(published in 1837)*. It is a limiting case of binomial distribution when the number of trials n is very large and p (the proportion of successes) is very small so that np (the average number of successes) is a finite constant ($= \lambda$, say).

Definition. *The discrete probability distribution obeying the probability law*

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x\,!},\ x = 0, 1, 2, 3, \ldots, \ldots, \ldots$$

is called a ***Poisson distribution*** *with parameter* λ.

The mean of Poisson distribution is given by

$$\bar{x} = \sum_{i=0}^{\infty} x_i \, P(x_i) = 0\,P(0) + 1\,P(1) + 2\,P(2) + 3\,P(3) + \ldots$$

$$= 0.e^{-\lambda} + 1 \cdot \frac{\lambda e^{-\lambda}}{1\,!} + 2 \cdot \frac{\lambda^2 e^{-\lambda}}{2\,!} + 3 \cdot \frac{\lambda^3 e^{-\lambda}}{3\,!} + \ldots = \lambda e^{-\lambda}\left(1 + \frac{\lambda}{1\,!} + \frac{\lambda^2}{2\,!} + \frac{\lambda^3}{3\,!} + \ldots\right) = \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda.$$

The variance of *Poisson distribution* is given by :

$$\textbf{Var (x)} = \sum_{i=1}^{\infty} x_i^2 \, P(x_i) - (\bar{x})^2$$

$$= \left[0 \cdot e^{-\lambda} + 1^2 \cdot \frac{\lambda e^{-\lambda}}{1\,!} + 2^2 \cdot \frac{\lambda^2 e^{-\lambda}}{2\,!} + 3^2 \cdot \frac{\lambda^3 e^{-\lambda}}{3\,!} + \ldots\right] - \lambda^2$$

$$= \left[\lambda e^{-\lambda} + \frac{2\lambda^2 e^{-\lambda}}{1\,!} + \frac{3\lambda^3 e^{-\lambda}}{2\,!} + \frac{4\lambda^4 e^{-\lambda}}{3\,!} + \ldots\right] - \lambda^2$$

$$= \left[\lambda e^{-\lambda} + \frac{(1+1)\,\lambda^2 e^{-\lambda}}{1\,!} + \frac{(1+2)\,\lambda^3 e^{-\lambda}}{2\,!} + \frac{(1+3)\,\lambda^4 e^{-\lambda}}{3\,!} + \ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1\,!} + \frac{\lambda^2}{2\,!} + \ldots\right] + \lambda^2 e^{-\lambda}\left[1 + \frac{\lambda}{1\,!} + \frac{\lambda^2}{2\,!} + \ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda} e^{\lambda} + \lambda^2 e^{-\lambda} e^{\lambda} - \lambda^2 = \lambda$$

$$= \textbf{Mean.}$$

15.5-1. Recurrence Formula For Poisson Distribution

We have, $\quad P(x) = \dfrac{e^{-\lambda} . \lambda^x}{x\,!}$ and $P(x+1) = \dfrac{e^{-\lambda} . \lambda^{x+1}}{(x+1)\,!}$.

$$\therefore \qquad \frac{P(x+1)}{P(x)} = \frac{\dfrac{e^{-\lambda} . \lambda^{x+1}}{(x+1)\,!}}{\dfrac{e^{-\lambda} . \lambda^x}{x\,!}} = \frac{e^{-\lambda} . \lambda^{x+1}}{e^{-\lambda} . \lambda^x} \times \frac{x\,!}{(x+1)\,!} = \frac{\lambda}{x+1}$$

Therefore, $\quad \boldsymbol{P(x+1) = \dfrac{\lambda}{x+1} P(x)}, \qquad$ where $x = 0, 1, 2, 3, \ldots,$

is called the *recurrence formula* for the Poisson distribution.

Important properties of Poisson distribution :

(*i*) As demonstrated above, the mean and variance of Poisson distribution coincide.

(*ii*) If X and Y are two independent Poisson random variables with mean λ and μ respectively, then $X.Y$ is also a Poisson random variable with mean $\lambda + \mu$. This is called the *additivity* property of Poisson random variables.

Q. 1. Define Poisson Distribution. Also, determine the mean and variance of the Poisson distribution.

2. Prove that the mean and variance of Poisson Distribution are same.

15.5-2. Illustrative Examples

Example 15. *The number of customers arriving at a facility for service between 10 A.M. and 11 A.M. is a random variable, say X_1, with Poisson distribution with mean 2. Similarly, the number of customers arriving between 11 A.M. and 12 noon, say X_2, has a Poisson distribution with mean 6. If X_1 and X_2 are independent, find the probability that more than 5 customers will come between 10 A.M. and 12 noon.*

Solution. By additive property, $X\ (= X_1 + X_2)$ possesses a Poisson distribution with mean $8 (= 2 + 6)$.

Therefore, the probability that there are x customers between 10 A.M. and 12 noon is given by

$$P(X=x)=\frac{e^{-8}\,8^x}{x\,!}\,,\ x=0,1,2,3,\ldots,\ldots$$

Hence the probability that more than 5 customers arrive is given by

$$P(X>5)=1-P(X\le 5)=1-\sum_{x=0}^{5}\frac{e^{-8}\,8^x}{x\,!}=1-0\cdot191=0\cdot809.$$

Example 16. *A box contains 200 tickets each bearing one of the numbers from 1 to 200, 20 tickets are drawn successively with replacement from the box. Find the probability that at most 4 tickets bear number divisible by 20.*

Solution. Here $n=20, p=P$ (divisible by 20 from 1 to 200) $=\dfrac{10}{200}=\dfrac{1}{20}\cdot$

$$Mean=np=20\times\frac{1}{20}=1.\ \text{Therefore, } mean=\lambda=1.$$

$$P(r)=\frac{e^{-\lambda}\lambda^r}{r\,!},\ P(0)=\frac{e^{-1}.(1)^0}{0\,!}=e^{-1}$$

Now, since $P(r+1)=\dfrac{\lambda}{r+1}P(r)$, we get

$$P(1)=\frac{1}{0+1}P(0)=1\cdot P(0)=1\cdot e^{-1}=e^{-1},\ P(2)=\frac{1}{1+1}P(1)=\frac{1}{2}\cdot P(1)=\frac{1}{2}\cdot e^{-1}=\frac{1}{2\,!}e^{-1},$$

$$P(3)=\frac{1}{2+1}P(2)=\frac{1}{3}\cdot\frac{1}{2\,!}e^{-1}=\frac{1}{3\,!}e^{-1},\ P(4)=\frac{1}{3+1}P(3)=\frac{1}{4}\cdot\frac{1}{3\,!}e^{-1}=\frac{1}{4\,!}e^{-1}.$$

$\therefore$ Reqd. prob. $=P(0)+P(1)+P(2)+P(3)+P(4)=e^{-1}+e^{-1}+\dfrac{1}{2\,!}e^{-1}+\dfrac{1}{3\,!}e^{-1}+\dfrac{1}{4\,!}e^{-1}$

$$=e^{-1}\left(2+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}\right)\cdot$$

Example 17. *If a random variable has a Poisson distribution such that* $P(1)=P(2)$, *find* $P(4)$.

(use $e^{-2}=0.1353$)

Solution. Since $P(r)=\dfrac{e^{-\lambda}\lambda^r}{r\,!}$, we get $P(1)=e^{-\lambda}.\lambda,\ P(2)=\dfrac{e^{-\lambda}\lambda^2}{2\,!}$

But, given that $P(1)=P(2)$. Therefore,

$$e^{-\lambda}.\lambda=\frac{e^{-\lambda}.\lambda^2}{2\,!}\Rightarrow\lambda=\frac{\lambda^2}{2}\ \text{or}\ \lambda=2.$$

Now, $$P(4)=\frac{e^{-2}.2^4}{4\,!}=\frac{0.1353\times16}{4\times3\times2\times1}=\frac{2}{3}(0.1353)=\frac{0.2706}{3}=0.0902.$$

Example 18. *If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals (a) exactly 3, (b) more than 2, individuals will suffer a bad reaction.* *(use* $e^{-2}=0.136$)

Solution. Here $p=0.001, n=2000$. Therefore, $\lambda=np=2000\times0.001=2$.

Since *Poisson distribution* is given by $P(r)=\dfrac{e^{-\lambda}\lambda^r}{r\,!},\ r=0,1,2,\ldots,$ we get

(*a*) $P(3)=\dfrac{e^{-2}.2^3}{3\,!}=\dfrac{4}{3}e^{-2}=\dfrac{4}{3}\times0.136=0.18.$

(*b*) P (more than 2) $=P(3)+P(4)+P(5)+\ldots=1-[P(0)+P(1)+P(2)]$

$$=1-\left[\frac{e^{-2}.2^0}{0\,!}+\frac{e^{-2}.2^1}{1\,!}+\frac{e^{-2}.2^2}{2\,!}\right]$$

$$=1-(1+2+2)e^{-2}=1-5\times0\cdot136=1-0.680=0.32.$$

Example 19. *If the variance of the poisson distribution is 2, find the distribution for r = 1, 2, 3, 4, and 5 from the recurrence relation of the poisson distribution (use $e^{-2} = 0.1353$).*

Solution. Mean of P.D. = Variance of P.D. = 2. Therefore, $\lambda = 2$.

Since $$P(r) = \frac{e^{-\lambda}\lambda^r}{r!}, \quad P(0) = \frac{e^{-2} \cdot 2^0}{0!} = e^{-2}.$$

Also, $$P(r+1) = \frac{\lambda}{r+1} P(r) = \frac{2}{r+1} P(r)$$

$\therefore$ $$P(1) = \frac{2}{0+1} \times P(0) = 2e^{-2} = 2 \times 0.1353 = 0.2706, \qquad (\because e^{-2} = 0.1353)$$

$$P(2) = \frac{2}{1+1} \times P(1) = \frac{2}{2} \times 0.2706 = 0.2706, \; P(3) = \frac{2}{2+1} \times P(2) = \frac{2}{3} \times 0{\cdot}2706 = 0.1804.$$

Similarly, $P(4)$, $P(5)$ can be obtained.

Example 20. *Using Poisson's distribution, find the probability that the ace of spades will be drawn from a pack of well-shuffled cards at least once in 104 consecutive trials.*

Solution. Given $P(r) = \frac{e^{-\lambda}\cdot\lambda^r}{r!}$, $r = 0, 1, 2, \ldots$, Also we have $\lambda = np = 104 \times \frac{1}{52} = 2$.

Therefore, prob. of drawing an ace of spades at least once

$$= 1 - P(0) = 1 - \frac{e^{-2} \cdot 2^0}{0!} = 1 - e^{-2} = 1 - 0.1353 = 0.8647.$$

15.6. THE GEOMETRIC DISTRIBUTION

There are two common versions of geometric distribution.

Definition 1. *A discrete random variable X is said to possess a* ***Geometric Distribution*** over *the range* $x = 1, 2, \ldots, \infty$, if it assumes only the non-negative values and obey the probability law

$$P(X = x) = p(1-p)^{x-1}, x = 1, 2, \ldots, \infty.$$

Then X is said to have the geometric distribution beginning at 1.

Definition 2. *If it is defined over the range $x = 0, 1, 2, \ldots, \infty$ and*

$$P(X = x) = \begin{cases} p(1-p)^x, & x = 0, 1, 2, \ldots, \infty \\ 0 \quad , & \text{otherwise} \end{cases}$$

then X is said to have the geometric distribution beginning at 0.

It is clear that one version is just a shifted version of the other, and other shifts can be done without altering the form of the distribution. Both of above versions appear in applications.

The expectation and variance for the geometric distribution beginning at 1 are, respectively.

$$E(X) = \frac{1}{p}, \quad V(X) = \frac{1-p}{p^2}.$$

In the other version, when the distribution begins at 0, the variance remains the same, but the expectation is $(1-p)/p$.

In either of the above two versions, X counts trials, so the geometric distribution is sometimes known as "*waiting time distribution*" in a queueing system.

Example 21. *A couple decides to have children until they have male child. If the probability of having a male child in their community is ⅓, find the expected number of children before the first male child is born.*

Solution. Obviously, the waiting time for a male child has a geometric distribution with $p = ⅓$. Consequently, $q = 1 - p = ⅔$. Hence, the expected number of children (*i.e.* mean) $= q/p = 2$. **Ans.**

15.7. THE NEGATIVE BINOMIAL DISTRIBUTION

Let X be a random variable defined over the range $x = r, r+1, \ldots \ldots, \infty$. Then X is said to obey negative binomial distribution if

$$P(x) = {}^{x-1}C_{r-1}\, p^r (1-p)^{x-r}, \quad x = r, r+1, \ldots \ldots, \infty$$

where r is an integer ≥ 1, and $0 \leq p \leq 1$. This distribution is also sometimes called the ***Pascal distribution.*** If $r = 1$, then this distribution reduces to the *geometric.*

The expectation and variance are given by $E(X) = \frac{r}{p}$, $V(X) = \frac{r(1-p)}{p^2}$.

The explanation of this distribution is just the extension of the *geometric.* Here X represents the number of the trial, in a sequence of independent Bernoulli trials on which the rth success occurs. So the *negative binomial distribution* is another waiting time distribution. Interpreting X in this manner suggests that the waiting time for the rth success must be sum of r waiting times for the one success. Since the trials are independent, this logic is true. In fact, the sum of r geometrically distributed random variables yields a random variable whose distribution is negative binomial with parameter r.

Table 15.1

Discrete Probability Distributions

	Name of Distribution	*Range*	*Parameters*	*PDF*	*Expectation*	*Variance*
1.	**Discrete Uniform**	$x = 1, 2, ..., N$	$N = 1, 2, ...$	$P(x) = 1/N$	$(N+1)/2$	$(N^2-1)/12$
2.	**Bernoulli**	$x = 0, 1$	$0 \leq p \leq 1$	$P(0) = 1-p, P(1) = p$	p	$p(1-p)$
3.	**Binomial**	$x = 0, 1, ..., n$	$n = 1, 2, ...$ $0 \leq p \leq 1$	$P(x) = {}^nC_x p^x (1-p)^{n-x}$	np	$np(1-p)$
4.	**Poisson**	$x = 0, 1, ..., \infty$	$\lambda > 0$	$P(x) = \frac{\lambda^x e^{-1}}{x!}$	λ	λ
5.	**Geometric**	$x = 1, 2, ..., \infty$	$0 \leq p \leq 1$	$P(x) = p(1-p)^{x-1}$	$1/p$	$(1-p)/p^2$
6.	**Negative Binomial (Pascal)**	$x = r, r+1, ... \infty$	$r = 1, 2, ...$ $0 \leq p \leq 1$	$P(x) = {}^{x-1}C_{r-1} p^r (1-p)^{x-r}$	r/p	$r(1-p)/p^2$

The negative binomial distribution is also sometimes used without any waiting-time interpretation, because of the simple reason that the parameters can be adjusted so as to fit a set of data. Under such circumstances, it may be desirable to have the range of X start at 0, instead of r. In this case, the suitable probability density function would be

$$P(x) = {}^{r+x-1}C_x p^r (1-p)^x \quad \text{for } x = 0, 1, \dots \dots, \infty$$

The variance will remain unchanged, but the expectation would become $r(1-p)/p$.

II-Continuous Probability Distributions

15.8. THE CONTINUOUS UNIFORM (RECTANGULAR OR HOMOGENEOUS) DISTRIBUTION

Let a continuous random variable X be restricted to a finite range, $a \leq x \leq b$, and be such that "no value is any more likely than any other." Then, X would be appropriately defined by the *continuous uniform distribution.* Obviously, it is the analog of the *discrete uniform distribution,* which restricted the random variable to a finite number of equally likely values. The statement "no value is more likely than any other" is some what vague, because, in fact, the probability of any one value for a continuous random variable is zero. A less intuitive but better statement would be "the probability that x falls within any interval in the range of X depends only on the width of the interval and not on its location".

In any case, this distribution is defined by its probability density function $f(x) = 1/(b-a)$, $a \leq x \leq b$.

The *expectation* is at the midpoint of the range, $E(X) = (a+b)/2$,

and the *variance* is given by $V(X) = (b-a)^2/2$.

Remember that this distribution is also known as ***rectangular distribution.***

15.9. THE NORMAL DISTRIBUTION

Normal distribution is due to *Demoivre* who first defined it in 1773. It is also associated with the names of *Laplace* and *Gauss* who discussed it at the close of the 18th century. After the name of *Gauss*, it is also sometimes, called the *Gaussian Distribution.*

Definition. *A continuous random variable X with the range space* $(-\infty, \infty)$ *having the probability density function*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{where } -\infty \leq x \leq \infty;\ -\infty \leq \mu \leq \infty;\ \sigma > 0,$$

is said to possess a Normal Distribution with parameters μ and σ, *denoted by* $N(\mu, \sigma^2)$. *Here* μ *is the mean and* σ *is the standard deviation of the normal distribution.*

If X is a normal random variable with mean μ and standard deviation σ, then the random variable $Z = (X - \mu)/\sigma$ is called the *standard normal variate* which has the normal distribution with mean 0 and S.D. unity, and is given by

$$f(Z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}z^2},\ -\infty \leq Z \leq \infty.$$

Normal Probability Curve

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},\ -\infty \leq x \leq \infty.$$

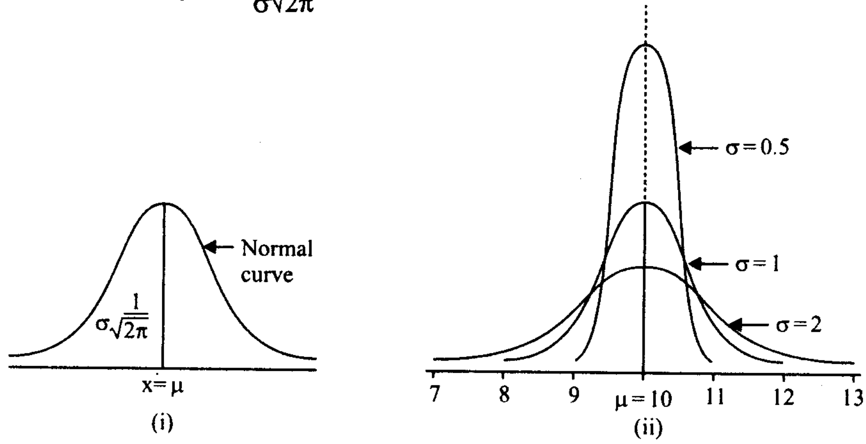

Fig. 15.1.

15.9-1. Properties of Normal Distribution Curve

1. The normal probability curve $y = f(x)$ is perfectly symmetrical about the line $x = \mu$ and is asymptotic to x-axis.
2. Normal probability curves with same mean and different variances differ in maximum ordinate. More the variance, lesser is the max. ordinate $\left(\frac{1}{\sigma\sqrt{2\pi}}\right)$. In Fig. 15.1 (*ii*), three normal curves are shown with same mean but having different standard deviations.
3. ***Mean, mode*** and ***median*** of normal distribution coincide at $x = \mu$.
4. The *mean* and *variance* are μ and σ^2 respectively.
5. The *mean deviation* about the *mean* of normal distribution is about 4/5 of its standard deviation.
6. The curve has two points of inflexion at $x = \mu \pm \sigma$. Both of these points are equidistant from the mean.
7. Total area under the curve and above x-axis from $-\infty$ to ∞ is unity.
   Area of normal curve between $\mu - \sigma$ and $\mu + \sigma$ is 68.27%.
   Area between $\mu - 2\sigma$ and $\mu + 2\sigma$ is 95.45%.
   Area between $\mu - 3\sigma$ and $\mu + 3\sigma$ is 99.73%.
8. The most probable limits for a normal variate are $\mu \pm 3\sigma$, *i.e.* $P[\mu - 3\sigma \leq X \leq \mu + 3\sigma] = 0.997$.
9. The sum and difference of independent normal variables also has a normal distribution. It is interesting to note that the normal density function is the most important and useful continuous probability density function. Also, besides arising in many practical stituations, it serves as a highly accurate approximation to other distributions.

15.9-2. Illustrative Examples

Example 22. *Find the area under the standard normal curve which lies :*
(i) to the right of $Z = -0.66$ *(ii) between* $Z = 1.25$ *and* $Z = 1.67$.

Solution. (*i*) Required area $= 1 - F(-0.66) = 1 - [1 - F(0.66)] = F(0.66) = 0.7454$.
(*ii*) Required area $= F(1.67) - F(1.25) = 0.9525 - 0.8944 = 0.0581$.

Example 23. *Find the area under the standard normal curve which lies :*
(*i*) *between* $Z = -0.90$ *and* $Z = -1.85$ (*ii*) *between* $Z = -0.90$ *and* $Z = 1.58$.

Solution. (*i*) Required Area $= F(-0.90) - F(-1.85) = [1 - F(0.90)] - [1 - F(1.85)]$
$= 1 - F(0.90) - 1 + F(1.85) = F(1.85) - F(0.90) = 0.9678 - 0.8159 = 0.1519$.
(*ii*) Required Area $= F(1.58) - F(-0.90) = 0.9429 - [1 - F(0.90)]$
$= 0.9429 - [1 - 0.8159] = 0.9429 - 0.1841 = 0.7588$.

Example 24. *It is known from the past experience that the number of telephone calls made daily in a certain community between 3 p.m. and 4 p.m. have a mean of 352 and a S.D. of 31. What percentage of the time will there be more than 400 telephone calls made in this community between 3 p.m. and 4 p.m. ?*

Solution. $\mu = 352,\ \sigma = 31,\ Z = \dfrac{X - \mu}{\sigma} = \dfrac{400 - 352}{31} = \dfrac{48}{31} = 1.55$.

Now, $F(1.55) = 0.9394$. $\therefore$ Probability $= 1 - F(1.55) = 1 - 0.9394 = 0.0606$.
$\therefore$ The required percentage of time $= (0.0606 \times 100) = 6.06\%$.

Example 25. *For a normal variate x with mean = 20 and variance = 16, find (i) the prob. that a value of x chosen at random lies between 12 and 24, (ii) the prob. that the value chosen at random lies between 16 and 18.*

Solution. Let $Z = \dfrac{X - \mu}{\sigma}$, $\mu = 20, \sigma = 4$. $\therefore$ $Z_1 = \dfrac{12 - 20}{4} = -2$, $Z_2 = \dfrac{24 - 20}{4} = 1$,

$\therefore$ $P(12 < x \le 24) = P(-2 \le Z \le 1) = P(-2 \le Z \le 0) + P(0 \le Z \le 1) = 0.4772 + 0.3413 = 0.8185$.

(*ii*) $Z_1 = \dfrac{16 - 20}{4} = -1$, $Z_2 = \dfrac{18 - 20}{4} = -0.5$.

$\therefore$ $P(16 \le x \le 18) = P(-1 \le Z \le -0.5) = P(-1 \le Z \le 0) - P(-0.5 \le Z \le 0)$
$= P(0 \le Z \le 1) - P(0 \le Z \le 0.5) = 0.3413 - 0.1915 = 0.1498$.

Example 26. *The lifetimes of certain kinds of electronic devices have a mean of 300 hours and a S.D. of 25 hours. Assuming that the distribution of these life times, which are measured to the nearest hour, can be approximated closely with a normal curve.*
(*i*) *Find the prob. that any one of these electronic devices will have a life time of more than 350 hours.*
(*ii*) *What percentage will have lifetimes of 300 hours or less ?*
(*iii*) *What percentage will have lifetimes from 220 or 260 hrs. ?*

Solution. Here $\mu = 300, \sigma = 25$. Let $Z = \dfrac{X - \mu}{\sigma}$. Here $X = 350$ $\therefore$ $Z = \dfrac{350 - 300}{25} = \dfrac{50}{25} = 2$. $\therefore F(2) = 0.9772$.

(*i*) Required prob. $= 1 - F(2) = 1 - 0.9772 = 0.0228$.

(*ii*) Here $Z = \dfrac{300 - 300}{25} = 0$, $F(0) = 0.5000$. $\therefore$ Required percentage $= 0.5000 \times 100 = 50\%$.

(*iii*) Here $Z_1 = \dfrac{220 - 300}{25} = -\dfrac{80}{25} = -3.2$, $Z_2 = \dfrac{260 - 300}{25} = -\dfrac{40}{25} = -1.6$.

Required prob. $= F(Z_2) - F(Z_1) = F(-1.6) - F(-3.2)$
$= \{1 - F(1.6)\} - \{1 - F(3.2)\} = (1 - 0.9452) - (1 - 0.9903) = 0.0548 - 0.0007 = 0.0541$.
$\therefore$ Required percentage $= 0.0541 \times 100 = 5.41\%$.

Example 27. *The marks obtained by a large group of students in a final examination in statistics have a mean of 58 and a S.D. of 8.5. Assuming that these marks are approximately normally distributed, what percentage of students can be expected to have obtained marks from 60 to 69, both, inclusive ?*

Solution. Here $\mu = 58$, $\sigma = 8.5$. and $Z = \dfrac{X - \mu}{\sigma}$

$\therefore \quad Z_1 = \frac{60-58}{8.5} = \frac{2}{8.5} = 0.24$ and $Z_2 = \frac{69-58}{8.5} = \frac{11}{8.5} = 1.29.$

$F(Z_2) - F(Z_1) = F(1.29) - F(0.24) = 0.9015 - 0.5948 = 0.3067.$

The percentage of students expected to have obtained marks from 60 to 69 = $0.3067 \times 100 = 30.67\%$.

Example 28. *In an intelligence test administered to 1000 students the average was 42 and standard deviation 24. Find (i) the number of students exceeding a score 50, (ii) the no. of students lying between 30 and 54.*

Solution. $\mu = 42, \sigma = 24$. Let $Z = (X - \mu)/\sigma, X = 50,$

$$\therefore \quad Z = \frac{50-42}{24} = \frac{8}{24} = \frac{1}{3} = 0.3333... = 0.33 \text{ (nearly)}$$

[$\because$ from tables $F(0.33) = 0.6293$]

$\therefore$ Area to the right of ordinate at $0.333 = 1 - F(0.33) = 1 - 0.6293 = 0.3707$

$\therefore$ Expected no. of students exceeding a score of $50 = 1000 \times 0.3707 = 370.7 = 371$ (nearly).

(*ii*) Now $X_1 = 30,\ X_2 = 54.$

$\therefore \quad Z_1 = \frac{30-42}{24} = -\frac{12}{24} = -0.5$ and $Z_2 = \frac{54-42}{24} = \frac{12}{24} = 0.5.$

Area to the right at $0.5 = 1 - F(0.5) = 1 - 0.6915 = 0.3085.$

Area to the left at $-0.5 = F(-0.5) = 1 - F(0.5) = 1 - 0.6915 = 0.3085.$

$\therefore \quad P(30 \le x \le 54) = 1 - (0.3085 + 0.3085) = 1 - 0.6170 = 0.3830.$

$\therefore$ The number of students having score between 30 and 54 = $1000 \times 0.3830 = 383$.

15.10. THE LOGNORMAL DISTRIBUTION

15.10-1. Definition

The distribution of a random variable whose natural logarithm follows a normal distribution is called ***lognormal distribution***. The lognormal density function is given by

$$f(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-(\log x - \mu)^2/2\sigma^2}$$

The range of the random variable is all $x > 0$. The parameters μ and σ may be define by

$$\mu = E(\log X),\ \sigma^2 = V(\log X).$$

15.10-2. The Mean and Variance

The mean and variance of X are respectively, given by $E(X) = e^{\mu + (1/2\sigma^2)}$, $V(X) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$.

While applying lognormal distribution practically, the best way is to transform it by taking its natural logarithm, say $Y = \log X$. In fact Y would be normally distributed and therefore can be easily handled.

For example, suppose we want to determine $P(a \le X \le b)$ where X is distributed lognormally. Now using the fact that Y is normal, we have

$$P(a \le X \le b) = P(\log a \le Y \le \log b) = \int_{\log a}^{\log b} \frac{1}{\sigma\sqrt{2\pi}} e^{-[(y-\mu)^2/2\sigma^2]}\, dy$$

which in turn would be evaluated by standardizing Y. We define $Z = (Y - \mu)/\sigma$, and

$$P(\log a \le Y \le \log b) = P\left(\frac{\log a - \mu}{\sigma} \le Z \le \frac{\log b - \mu}{\sigma}\right) = \int_{(\log a - \mu)/\sigma}^{(\log b - \mu)/\sigma} \phi(z)\, dz$$

We have seen that the *normal distribution* arises from the sum of independent random variables. But, in contrast to this, the lognormal distribution arises from the product of many independent non-negative random variables.

15.10-3. Applications of Lognormal Distribution

The lognormal distribution has been used to describe lifetimes of electrical and mechanical systems, abundance of species of animals, incubation periods of infectious diseases, concentration of chemical elements in geological materials, and many other random phenomena occurring in both the social and natural sciences.

The usefulness of the lognormal distribution as demonstrated above immediately suggests the more general concept of fitting the well-known distribution to *transformed* data rather than to the data itself. The normal distribution together with its logarithmic transformation is just one possible combination. Trigonometric, exponential, square root, and quadratic transformations can also be applied to the raw data in order to find a simple distribution to explain them.

15.11. THE NEGATIVE EXPONENTIAL DISTRIBUTION

15.11-1. Definition

A continuous random variable X is said to have *Negative Exponential Distribution* with parameter μ if it assumes only real positive values and its probability density function is given by

$$f(x) = \begin{cases} ae^{-ax}, & \text{if } x > 0 \\ 0 \ , & \text{otherwise} \end{cases}$$

where $a > 0$ is the prescribed parameter.

15.11-2. The Mean and Variance

The mean and variance of this distribution are $1/a$ and $1/a^2$ respectively.

This distribution is sometimes simply known as *exponential distribution* only.

15.11-3. Properties of Exponential Distribution

1. The *exponential distribution* is the continuous *analog* of the geometric distribution.
   For example, if the geometric random variable represents number of trials before the first failure occurs, its equivalence in the exponential case would be the waiting time for failure. In fact, as $p \to 0$, and *inter-trial time* $\to 0$, the *geometric distribution* $\to$ *exponential distribution.*
2. There exists an important relationship between the *Poisson distribution* and the *exponential distribution.* If the *Poisson distribution* represents the number of failure per unit time, the *exponential distribution* will describe the time between two successive failures.

15.12. THE ERLANG DISTRIBUTION

15.12-1. Definition

A continuous random variable X defined over the range $x \geq 0$ is said to possess *Erlang distribution* if its probability density function is given by

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)\,!}, \ x \geq 0,$$

where the parameter $\lambda > 0$ and r is in integer ≥ 1.

In the case of $r = 1$, the density function immediately reduces to that of a negative exponential distribution.

Hence the Erlang distribution is the generalization of the negative exponential random variables, each having the parameter λ, then the sum of these random variables will be Erlang distributed with parameters λ and r. In case, each of the negative exponential random variables is a waiting time, then Erlang random variable can be considered as the time until the rth event.

15.12-2. The mean and Variance

The expectation can be easily determined as the sum of the expectations of the negative exponential random variables given by $E(X) = r/\lambda$, and the variance can also be obtained by following a similar logic as $V(X) = r/\lambda^2$.

15.12-3. Applications of Erlang Distribution

As discussed above, the Erlang distribution can be used as a waiting time for the rth event. This distribution is also often considered to fit empirical data in queueing, reliability, replacement, and inventory control applications. In this case, r has no physical meaning, it is simply a parameter which can be adjusted to get a better fit.

15.13. THE GAMMA DISTRIBUTION

15.13-1. Definition

A continuous random variable X defined over the range $x \geq 0$ is said to be *gamma* distributed if the probability density function is of the form

$$f(x) = \frac{\lambda^r x^{r-1}}{\Gamma(r)} e^{-\lambda x}, x \geq 0,$$

where both r and λ are positive, and $\Gamma(r)$ [to be read as 'gamma r'] is the gamma function, defined by

$$\Gamma(r) = \int_0^{\infty} x^{r-1} e^{-x} \, dx.$$

Since the gamma function is tabulated, the value of $\Gamma(r)$, thus being a constant, can be easily obtained for the given value of r.

If r has an integer value, then $\Gamma(r) = (r-1)\,!$. In such a case, the gamma distribution immediately reduces to the Erlang. In one way, the gamma can be thought of as a generalization of the Erlang in which r need not have an integer value. Due to this generalization we have some what more flexibility in fitting empirical data.

15.13-2. The Mean and Variance

The expectation and variance of a gamma distributed random variable X are, respectively,

$$E(X) = r/\lambda \quad \text{and} \quad V(X) = r/\lambda^2 \cdot$$

15.13-3. A Particular Case

If $\lambda = 1/2$ and $r = v/2$, where v is an integer, the Γ density function becomes

$$f(x) = \frac{1}{2^{v/2}\,\Gamma(v/2)} x^{(v-2)/2} e^{-(x/2)}, x \geq 0.$$

This particular form is generally known as ***chi-square density function,*** and the single parameter v is called the degrees of freedom associated with the chi-square random variable. A chi-square random variable with v degrees of freedom arises when a set of v independent standard normal random variables are each squared and then summed-up.

15.13-4. Applications of Gamma Distribution

The gamma distribution has many applications in statistical hypothesis testing, but is not often used as a descriptive distribution in modeling applications. For the latter purpose, the Erlang distribution is a much more useful and special case of gamma function.

15.14. THE WEIBULL DISTRIBUTION

15.14-1. Definition

Let X be a continuous random variable defined over the range $x \geq 0$. It is said to have a *Weibull distribution* if its density function is of the form $f(x) = \lambda\beta\,(\lambda x)^{\beta-1}\, e^{-(\lambda x)^{\beta}}, x \geq 0.$
where λ and β both are positive constants. When $\beta = 1$, this density function at once reduces to that of the *negative exponential.* Thus *Weibull distribution* is the generalization of the *negative exponential.*

15.14-2. The mean and Variance

The mean and variance of a *Weibull* distributed random variable are, respectively, given by

$$E(X) = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\beta}\right) \quad \text{and} \quad V(X) = \frac{1}{\lambda^2}\left\{\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2\right\}$$

The gamma function $\Gamma(\ )$ occurring in these expressions can be found tabulated in many books of mathematical tables.

15.14-3. Special Cases

Since the *Erlang* family generalizes the negative exponential, and the gamma in turn generalizes the Erlang, we may think that the *Weibull* and *gamma distributions* are the same, or that one is the special case of the other. However, there are gamma distributions that are not Weibull, and there are Weibull distributions that are not gamma. So these two families are definitely distinct inspite of the fact that they each generalize the negative exponential family.

15.14-4. Applications

The Weibull distribution has been found to be useful for describing waiting times and lifetimes in reliability applications. When a system consists of a large number of parts, each of which has a lifetime distribution of its own (independent of others) and when the system fails as soon as any one of the parts does, then the lifetime of the system is the minimum of the lifetimes of its parts. Under such circumstances, there is theoretical justification for expecting a *Weibull distribution* to give a close approximation to the lifetime distribution of the system.

15.15. THE BETA DISTRIBUTION

15.15-1. Definition

A continuous random variable X defined over the range $0 \le x \le 1$ is said to have *beta distribution* if its density function is given by

$$f(x) = \frac{x^{r-1}(1-x)^{s-1}}{B(r, s)}, \; 0 \le x \le 1,$$

where $B(r, s)$ is the beta function, which is tabulated directly or may be found from tables of the gamma function from the relation

$$B(r, s) = \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)}.$$

15.15-2. The Mean and Variance

The mean and variance of X are, respectively, given by

$$E(X) = \frac{r}{r+s}, \; V(X) = \frac{rs}{(r+s)^2\,(r+s+1)}.$$

These relations can be inverted to find interpretations of the parameters r and s in terms of the mean and variance.

Table 15.2 : Continuous Probability Distributions

	Name of Distribution	Range	Parameter	PDF $[f(x)]$	Expectation	Variance
1.	Continuous Uniform	$a \le x \le b$	a, b	$1/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
2.	Normal	$-\infty < x < \infty$	μ, $\sigma^2 > 0$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-[(x-\mu)^2/2\sigma^2]}$	μ	σ^2
3.	Lognormal	$x \ge 0$	μ, $\sigma^2 > 0$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-[\log(x-\mu)^2/2\sigma^2]}$	$e^{\mu+(1/2\sigma^2)}$	$e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$
4.	Negative Exponential	$x \ge 0$	$\lambda > 0$	$\lambda e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$

Continued →

5.	(*i*) Gamma (*ii*) Erlang when r is an integer (*iii*) χ^2 when $\lambda = 1/2$, $r = \nu/2$.	$x \geq 0$	$\lambda > 0$ $r > 0$	$\frac{\lambda^r x^{r-1}}{\Gamma(r)} e^{-\lambda x}$	r/λ	r/λ^2
6.	Weibull	$x \geq 0$	$\lambda > 0$ $\beta > 0$	$\lambda\beta (\lambda x)^{\beta - 1} e^{-(\lambda x)^\beta}$	$\frac{1}{\lambda}\Gamma\left(1+\frac{1}{\beta}\right)$	$\frac{1}{\lambda^2}\left\{\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}$
7.	Beta	$0 \leq x \leq 1$	$r > 0$ $s > 0$	$\frac{x^{r-1}(1-x)^{s-1}}{B(r, s)}$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$

$$r = E(X)\left[\frac{E(X)\,[1-E(X)]}{V(X)} - 1\right], \quad s = [1-E(X)]\left[\frac{E(X)\,[1-E(X)]}{V(X)} - 1\right]$$

whenever it is desired to define a beta distributed random variable, say Y, having a range $a \leq Y \leq b$, the easiest way will be to transform Y according to the relation $X = (Y-a)/(b-a)$, which immediately gives us a beta-distributed random variable, defined over the range $0 \leq X \leq 1$, and thus has the density function given above.

15.15-3. Applications

The beta distribution is commonly used to describe the random variables which are not uniformly distributed, but whose possible values lie in a restricted interval of numbers. The range is sometimes restricted by the very nature of the random variable, for example, when it represents a fraction or a percentage. Other times, the range is restricted by choice. In PERT and CPM techniques of project sheduling networks, the durations of activities are often assumed to follow a beta-distribution, where the minimum and maximum possible times are supplied by persons who are familiar with the activity.

III - Selection of Appropriate Distribution to Data

15.16. HOW TO FIT DISTRIBUTIONS TO DATA

To describe a particular random phenomenon there is a necessity of selecting an appropriate distribution. In order to fulfill this necessity. We must first attempt to acquire data representing a large number of independent samples of random variable under consideration. But, sometimes, the collection of data may be economically infeasible or even physically impossible. Under such circumstances, there may exist theoretical justification to rely on a certain distribution family which is appropriate. For example,

For example, (*i*) if the phenomenon can be thought of as the number of successes in a sequence of independent Bernoulli trials, then *binomial distribution* will be appropriate; or

(*ii*) if the phenomenon can be thought of as consisting of the sum of a large number of independent random variables, the *central limit theorem will suggest us the normal distribution; and*

(*iii*) in other cases, the selection of distribution depends upon the need for particular properties, e.g. the *beta distribution* in PERT and CPM applications and the *negative exponential distribution* in Markov process models are selected reasons that have little to do with observed data.

15.17. MAIN STEPS OF FITTING DISTRIBUTION TO DATA

Step 1. First, we must have real world data to provide assurance that the distribution selected really does describe the real-world phenomenon. Since it is difficult to observe any pattern in a raw list of values, we should ordinarily plot a histogram to identify an appropriate distribution.

Step 2. The next step, requires a familiarity with the characteristics of various distribution families, in order to decide whether there is any hope of adjusting the parameters to get a probability density function that looks like the histogram.

Step 3. After tentatively selecting a type of distribution, the values of the parameters are set to fix the distribution within the family. For example, the parameter λ in a Poisson distribution is estimated by the sample mean, and μ and σ^2 in normal distribution are estimated by the sample mean and sample variance, respectively.

Step 4. After adjusting the parameters to provide the best fit to the data, which is provided by a selected distribution, we must see whether the fit is good enough. Methods of testing are provided in many introductory books on statistics.

Step 5. Finally, it is well to keep in mind that no amount of data can confirm absolutely that we have selected the correct distribution. So there is no escape from having to make assumptions and hence there is no need for a model to represent its real world referent perfectly.

EXAMINATION PROBLEMS

1. What is a frequency distribution and why is it important in statistics ?
2. Find the derivative of $y = e^{-t^2}$ and sketch the curve. Discuss the relationship of this curve to that of the normal distribution.
3. Distinguish between discrete and continuous probability distributions. Explain why the normal distribution is considered the most important distribution in probability and statistics.
4. Verify the sum of all the probabilities for a Binomial distribution as well as Poisson distribution turns out to be unity.
5. Give an example of a non-Poisson distribution which has the same mean and variance. **[Hint.** Consider uniform distribution for a suitable n.]
6. What is a normal distribution ? Highlight its chief uses in Operations Research.
7. Write down the density function of a normal distribution and discuss its important properties.
8. Verify that the exponential distribution is a special case of *Erlang* distribution when $n = 1$.
9. Illustrate some situations where Poisson distribution may be successfully employed.
10. If X has a Poisson distribution with parameter α, then show that $F(X) = \alpha$ and $V(X) = \alpha$. Further, show that the Poisson distribution is a limiting form of the Binomial distribution. **[I.A.S. (Main) 86]**
11. Find the binomial distribution whose mean is 10 and standard deviation is $2\sqrt{2}$.
12. Bring out the fallacy, if any in the following statement. The mean of a binomial distribution is 15 and its S–D is 5.
13. Ten per cent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random, exactly two will be defective by using
    (i) the bonomial distribution (ii) the Poisson approximation to the binomial distribution. **[IAS (Main) 94]**
14. Among the 300 employees of a company 240 are union members while the others are not. If 8 of the employees are to be chosen to serve on a committee which administers the pension fund, find the prob. that 5 of them will be union member while the others are not, using (i) hypergeometric distribution and (ii) binomial distribution. **[IAS (Main)96]**
15. A die is tossed. Let x denote twice the number appearing and Y denote 1 or 3, depending on whether an odd or an even number appears. Find the distribution, expectation and variance of X, Y and $X + Y$. **[IAS (Main)98]**
16. Let x_1 and x_2 have independent Gamma distributions with parameters α, θ and β, θ respectively. Let $Y_1 = X_1/(X_1 + X_2)$ and $Y_2 = X_1 + X_2$. Find p.d.f., $g(y_1, y_2)$ of Y_1 and Y_2. Show that Y_1 has a Beta p.d.f. with parameters α and β. **[IAS (Main) 98]**
17. (a) (i) If X is $N(3, 16)$, find $P(4 \le X \le 8)$, $P(0 \le X \le 5)$ and $P(-2 \le X \le 1)$.
    (ii) If X is $N(25, 36)$, find the constant C such that $P(|X - 25| \le C) = 0{\cdot}9544$.
    [Ans. (i) 0·2954, 0·4649, 0·2029 (ii) $C = 12$]. **[IAS (Main) 1998]**
18. If X is a $N(0, 1)$, find the probability density function of $|X|$.
    [Ans. $f(y) = \sqrt{2/\pi}\, e^{-x/2}$, $x \ge 0$] **[IAS (Main) 2000]**

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16

MARKOV ANALYSIS

16.1. INTRODUCTION

Markov process (or *chain*) is a stochastic (or random) process which has the property that the probability of transition from a given state to any future state depends only on the present state and not on the manner in which it was reached.

The nature of Markov processes can be easily understood by considering the situation of a Car-Rental company. Suppose a Car-Rental company is running agencies in different cities. A car sent to one city may return to any city where the company's agency is available. If this situation is considered as a *Markov process*, then the different rental cities would be the *states*. A particular transition probability p_{ij} would be the probability that a car rented to city i would return to city j, where j may be equal to i. The mathematical structure of this problem is to determine expected long term fraction of cars at each city and the mean number of trips a car would make starting from city i, before returning to that location.

Markov process is widely used in examining and predicting the behaviour of consumers in terms of their brand loyalty and their switching patterns to other brands.

Markov processes are also used in the study of equipment maintenance and failure problems analysing accounts receivable that will ultimately become bad debts. It is also used to study the stock market price movements.

16.2. STOCHASTIC (RANDOM) PROCESS

Definition. A *stochastic* (or *random*) process is defined as a family of random variables $\{X(t_n) : n = 1, 2, 3, \ldots\}$. The random variable $X(t)$ stands for the observation at time t. The number of states n may be *finite* or *infinite* ending upon the time range.

For example, s consider the poisson distribution

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n\,!}, n = 1, 2, 3, \ldots .$$

This distribution represents a stochastic (or random) process with infinite number of states. In this example, the random variable n denotes the number of occurrences between the time interval 0 and t (assuming that the system starts at 0 times). Thus the states of the system at any time t are given by n = 0, 1, 2, ...

16.3. MARKOV PROCESS

Definition. *A stochastic* (or *random*) *system is called a* ***Markov process*** *if the occurrence of a future state depends on the immediately preceding state and only on it.*

Thus if $t_0 < t_1 < \ldots < t_n$ represents the points in time scale then the family of random variables $\{X(t_n)\}$ is said to be a *Markov* process provided it holds the *Markovian* property :

$$P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, \ldots, X(t_0) = x_0\} = P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}$$

for all $X(t_0), X(t_1), \ldots, X(t_n)$.

Markov process is a sequence of n experiments in which each experiment has n possible outcomes $x_1, x_2, \ldots x_n$. Each individual outcome is called a *state* and the probability (that a particular outcome occurs) depends only on the probability of the outcome of the preceding experiment. The simplest of the *Markov* processes is *discrete* and *constant* over time. It is used when the sequence of experiments is completely described in terms of its states (possible outcomes). There is a finite set of states numbered 1, 2, ..., n, and this process can be only in one state at a prescribed time. A system is said to *discrete* in time if it is examined at regular intervals, *e.g. daily, weekly, monthly,* or *yearly.*

16.4. TRANSITION PROBABILITY

Definition . *The probability of moving from one state to another or remaining in the same state during a single time period is called the* ***transition probability.***

Mathematically, the probability

$$p_{x_{n-1}, x_n} = P\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}$$

is called the *transition probability.*

This represents the conditional probability of the system which is now in state x_n at time t_n, provided that it was previously in state x_{n-1} at time t_{n-1}. Sometimes this probability is known as *one step* transition probability, because it describes the system during the time interval (t_{n-1}, t_n).

Since each time a new result or outcome occur, the process is said to have *stepped* or *incremented* one step. Each step represents a time period or any other condition which would result in another possible outcome. The symbol n is used to indicate the number of steps or increments. For example, if $n = 0$, then it represents the initial state.

16.5. TRANSITION PROBABILITY MATRIX

The transition probabilities can be arranged in a matrix form and such a matrix is called a ***one-step transition probability matrix,*** denoted by

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

The matrix P is a squared matrix whose each element is non-negative and sum of the elements of each row is unity *i.e.,*

$$\sum_{j=1}^{m} p_{ij} = 1; \quad i = 1, 2, ..., m, \text{ and } 0 \le p_{ij} \le 1.$$

In general, any matrix P, whose elements are non-negative and sum of the elements either in each row or column is unity, is called a ***transition matrix*** or a ***probability matrix.*** Thus a transition matrix is a square stochastic matrix (since number of rows is equal to the number of columns in the matrix) and therefore, it gives the complete description of the Markov process.

Diagrammatic Representation of Transition Probabilities :

The transition probabilities can also be represented by two types of diagrams :

(1) Transition Diagram. Transition diagram shows the transition probabilities or shifts that can occur in any particular situation. Such a diagram is given in Fig. 16.1. :

The arrows from each state indicate the possible states to which a process can move from the given state. The matrix of transition probabilities which corresponds to above diagram is as given below :

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & ½ & ½ \\ ⅓ & 0 & ⅔ \end{bmatrix} \end{array}$$

A *zero* element in the above matrix indicates that the transition is not possible.

(2) Probability tree diagram. As the name implies, this diagram emphasizes the probabilities and their movement from one step to another step, alongwith all possible branches or paths that may connect the outcomes over a period of time. Tree diagram can be explained by the following example.

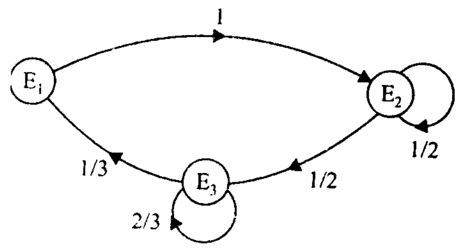


Fig. 16.1

Example 1. *Two manufacturers A and B are competing with each other in a restricted market. Over the years, A's customers have exhibited a high degree of loyalty as measured by the fact that customers using A's product 80% of time. Also former customers purchasing the product from B have switched back to A's 60% of time.*

(a) Construct and interpret the state trasition matrix in terms of (i) retention and loss (ii) retention and gain.

(b) Calculate the probability of a customer purchasing A's product at the end of the second period.

Solution. **(a)** The transition probabilties can be arranged in a matrix form as shown below.

$$\text{Present } P = \textit{Purchase } (n=0) \quad \begin{array}{c} \\ A \\ B \end{array} \overset{\text{Next purchase } (n=1)}{\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \left[\begin{array}{cc} 0.80 & 0.20 \\ 0.60 & 0.40 \end{array}\right] \end{array}} \; \textit{Retention} \text{ and } \textit{Gains} \downarrow$$

$$- \text{Retention and Losses} \rightarrow$$

Obviously, the probability of a customer's purchase at the next step $n = 1$ (next purchase depends upon the product which a customer is having at present ($n = 0$). Each probability in the above matrix must therefore be a conditional probability for passing from one state to another.

Mathematically, conditional probabilities in the above matrix can be stated as

(*i*) $P\,(A_o \mid A_1) = p_{11} = 0.80.$

This indicates that the probability that the customers now using A's product at $n = 0$ (present purchase) will again purchase A's product at $n = 1$ (next purchase) is 0.80. This means ***retention*** to A 's product.

(*ii*) $p\,(B_0 \mid A_1) = p_{21} = 0.60.$

This indicates that probability that the customer now using B's product at $n = 0$ (present purchase) will purchase A's product at $n = 1$ (next purchase) is 0.60. This means ***loss*** to B's product.

(*iii*) $P\,(A_0 \mid B_1) = p_{12} = 0.20.$

This indicates that the probability that the customer now using A 's product at $n = 0$ (present purchase) will purchase B's product at $n = 1$ (next purchase) is 0.20. This means ***loss*** to A 's product.

(*iv*) $P\,(B_0 \mid B_1) = p_{22} = 0.40.$

This indicates that the probability that the customer now using B's product at $n = 1$ (present purchase) will purchase B's product at $n = 1$ (next purchase)is 0.20. This means ***retention*** to B 's product.

(b) The transition probabilities can be represented by two types of diagrams :

(*i*) transition diagram as shown in *Fig.* 16.2, and (*ii*) probability tree diagram as show in *Fig.* 16.3.

In Fig 16.2, nodes indicate ***states*** and arrows represent the ***transition probabilities*** between states.

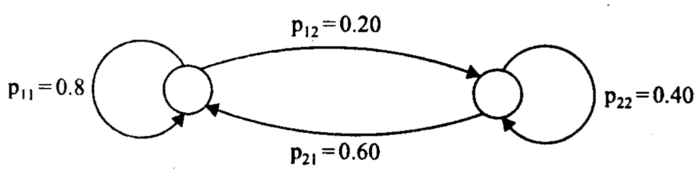


Fig. 16.2

Probability Tree Diagram

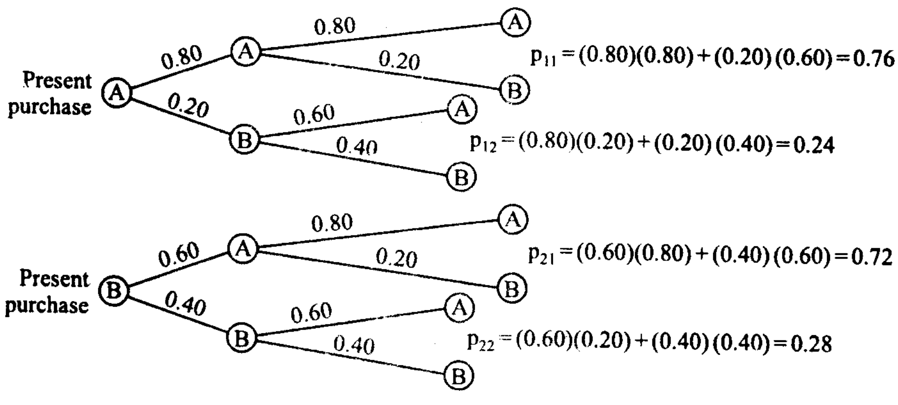


Fig. 16.3

Probability Computations :

If we begin with a customer's purchase of A 's product in the state E_1, at $n = 0$, then $P_1(0) = 1$ and $P_2(0) = 0$, so

$$R(0) = (1 \quad 0).$$

After the first transition, the row vector of state probabilities $R(1)$ which describes all possible outcomes at $n = 1$ is given by

$$R(1) = R(0)\,P = (1 \quad 0)\begin{pmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{pmatrix} = (0.80 \quad 0.20)$$

This means that if the present state is E_1, the probability (that the present state is E_1) is $P_1(1) = 0.80$, and (that the next state is E_2) is $p_2(1) = 0.20$. In other words, the probability of a customer using A's product at the end of step 1 is 80% and there are 20% chances that the customer will switch over to B's product at the end of step 1.

The probability that a customer using A's product in the state E_1 at $n = 0$ also uses A's product in the state E_2 at $n = 2$, can be obtained by calculating state row vector $R(2)$ of state probabilities which describes all possible outcomes in step $n = 2$.

$$R(2) = R(1)\,P = (0.80 \quad 0.20)\begin{pmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{pmatrix} = (0.76 \quad 0.24)$$

This indicates that if the present state is E_1 at $n = 0$, 2-step later (*i.e.*, $n = 2$), the probability of being in state E_1 is $p_1(2) = 0.76$ and in state E_2 is $p_2(2) = 0.24$. Thus the probability of A's product after the end of 2-step is 76% and that of B's product is 24%.

In a similar manner, if the present state is E_2, then $R(2) = (0.72 \quad 0.28)$ as obtained earlier.

16.6. FIRST ORDER AND HIGHER ORDER *MARKOV* PROCESS

The ***first order Markov process*** is based on the following three assumptions :

(i) The set of possible outcomes is *finite*.

(ii) The probability of the next outcome (state) depends only on the immediately preceding outcome.

(iii) The transition probabilities are constant over time.

The ***second order Markov process*** assumes that the probability of the next outcome (state) may depend on the two previous outcomes. Likewise, a ***third order Markov process*** asumes that the probability of the next outcome (state) can be calculated by obtaining and taking account of the outcomes of the past three outcomes.

But, in this chapter, we shall dicuss only *first order* Markov process.

16.7. *n*-STEP TRANSITION PROBABILITIES

Suppose the system which occupies state E_i at time $t = 0$, then we may be interested in finding out the probability that the system moves to state E_j at time $t = n$ (these time periods are sometimes referred to as number of steps). If the n-step transition probability is denoted by $p_{ij}^{(n)}$, then these transition probabilities can be represented in matrix form as given below.

$$P^{(n)} = \begin{array}{c} \\ E_1 \\ E_2 \\ \vdots \\ E_m \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_m \end{array} \\ \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} & \cdots & p_{1m}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & \cdots & p_{2m}^{(n)} \\ \vdots & \vdots & & \vdots \\ p_{m1}^{(n)} & p_{m2}^{(n)} & \cdots & p_{mm}^{(n)} \end{bmatrix} \end{array}$$

Here $p_{21}^{(n)}$, for example, means the probability that the system which occupies state E_2 will move to state E_1 after n steps.

Let $p_i(n)$ be the probability that the system occupies state E_i will move to state E_j in one transition. It should be noted that the transition probability p_{ij} is independent of time whereas the ***absolute probability*** $\boldsymbol{p_i(n)}$ depends on time. If the number of possible states be m, then

$$\sum_{i=1}^{m} p_i(n) = 1, \text{ and } \sum_{j=1}^{m} p_{ij} = 1 \text{ for all } i.$$

If all the state probabilities are known at time $t = n$, then the state probabilities at time $t = n + 1$ can be determined by the equation :

$$p_j(n+1) = \sum_{i=1}^{m} p_i(n)\, p_{ij}\ ;\ n = 0, 1, 2, ..$$

In other words, the probability of being in state E_j at time $t = n + 1$ is equal to the probability of being in state E_i at time $t = n$ multiplied by the probability of a transition from state E_i to state E_j for all values of i.

To make the procedure more clear, we may rewrite the equations for each state probability at time $t = n + 1$ as follows :

$$\begin{aligned} p_1(n+1) &= p_1(n)\, p_{11} + p_2(n)\, p_{21} + \ldots + p_m(n)\, p_{m1} \\ p_2(n+1) &= p_1(n)\, p_{12} + p_2(n)\, p_{22} + \ldots + p_m(n)\, p_{m2} \\ &\vdots \\ p_m(n+1) &= p_1(n)\, p_{1m} + p_2(n)\, p_{2m} + \ldots + p_m(n)\, p_{mm}. \end{aligned}$$

This system of equations can be written in matrix form as

$$R(n+1) = R(n)\, P, \qquad ...(16.1)$$

where $R(n + 1)$ is the row vector of state probabilities at time $t = n + 1$, $R(n)$ is the row vector of state probabilities at time $t = n$, and P is the matrix of transition probabilities.

If the state probabilities at time $t = 0$ are known, these can be found at any time by solving the matrix equation (16.1). that is,

$$R(1) = R(0)P,\ R(2) = R(1)P = R(0)P^2,\ R(3) = R(2)P = R(0)P^3 \ldots\ldots\ldots, R(n) = R(n-1)P = R(0)P^n.$$

Q. A *Markov* chain with three states α, β and γ defined by the transition matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

Taking the initial state to be α, determine the *n*th step transition probability $p_{i\alpha}^{(n)}$ and the sbsolute probabilities $p_j^{(n)}$.

16.8. MARKOV CHAIN

Let $p_j^{(0)}$ $(j = 0, 1, 2, ...)$ be the absolute probability such that the system be in state E_j at time t_0, where $E_j(j = 0, 1, 2, ...)$ denote the exhaustive and mutually exclusive outcomes (states) of a system at any time. Also it is assumed that the system is *Markovian.*

We now define $\quad p_{ij} = P\{X(t_n) = j \mid X(t_{n-1}) = i\}$

as the one-step transition probability of going from state i at time t_{n-1} to state j at time t_n. It is also assumed here that these probabilities from state E_i to state E_j $(i = 0, 1, 2, ...; j = 0, 1, 2, \ldots)$ are expressed in the matrix form as below :

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

This matrix P is known as *stochastic matrix* or *homogeneous matrix.* The probabilities p_{ij} must satisfy the boundary conditions : $\Sigma\, p_{ij} = 1$, for all i, and $p_{ij} \geq 0$ for all i and j

Definition. *Markov chain.* *The transition matrix P as defined above together with the initial probabilities* $\{p_j^{(o)}\}$ *associated with the states* $E_j(j = 0, 1, 2, \ldots)$ *completely define a* ***Markov chain.***

The *Markov* chains are of two types : (i) ergodic, (ii) regular.

An ***ergodic Markov*** chain has the property that it is possible to pass from one state to another in a finite number of steps, regardless of present state.

A special type of *ergodic Markov chain* is the ***regular Markov chain.***

A *regular Markov* chain is defined as a chain having a transition matrix P such that for some hour of P it has only non-zero positive probability values.

Thus all regular chains must be *ergodic* chains. The easiest way to 'check if an *ergodic* chain is *regular*' is to continue squaring the transition matrix P until all zeros are removed.

Example 2. *Determine if the following transition matrix is ergodic Markov chain.*

$$\text{Present States}\;\begin{array}{c|c} & \text{Future States} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1/2 & 1/4 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix} \end{array}$$

Solution. Here we must check that it is possible to go from every present state to all other states.

We observe that from state 1, it is possible to go directly to every other state except state 3. For state 3, it is possible to go from state 1 to state 2 to state 3. Therefore it is possible to go from state 1 to any other state. Similarly, from state 2, it is possible to go to state 3 or state 4, then from state 3 to state 1, or from state 4 to state 3 to state 1. Also, from state 3 it is possible to go directly to state 1. Finally, from state 4, it is possible to go to state 3, then from state 3 to state 1. Hence above transition matrix is an *ergodic Markov chain.*

Example 3. *Test the following transition matrix to see if the Markov chain is regular and ergodic where x is some positive p_{ij} value.*

$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix} \end{array}$$

Solution. We compute :

$$P^2 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array} \qquad P^4 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array} \qquad P^8 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array}$$

From this, we observe that P raised to an even-number power gives the result as above, while P raised to an odd-number power will give the original matrix. Since all the elements are not non-zero positive elements, the above matrix is not regular. But, it is an *ergodic* since it is possible to go from state 1 to state 2 or state 3 from state 2 to state 1 or state 4. From state 2 to state 1. From state 3 to state 1. From state 4 to state 2 or state 1.

16.9. STEADY STATE (EQUILIBRIUM) CONDITION

The determination of ***steady state*** conditions in a *regular ergodic Markov chain* can be accomplished most readily by computing *pn* for larger values of n. Therefore, there is a limiting probability that the system will reach to steady state equilibrium in a finite number of transitions. This can be generalized in the following manner.

In equation (16.1), if n becomes very large, the values p_{ij} tend to fixed limits and each probability vector $R(n)$ approaches a constant value *i.e.*, $R(n+1) = R(n) = R$.

Thus, taking limits,

$$\lim_{n \to \infty} R(n+1) = \lim_{n \to \infty} R(n)P \quad \text{or} \quad R = RP.$$

Therfore, as $n \to \infty$, $R(n)$ becomes constant (*i.e.* independent of time) and then the system is said to have reached to a steady state equilibrium.

Example 4. *A manufacturing company has a certain piece of equipment that is inspected at the end of each day and classified as* ***just overhauled good, fair*** *or* ***inoperative.*** *If the item is inoperative, it is overhauled, a procedure that takes one day. Let us denote the four classifications as states 1,2,3 and 4* *respectively. Assume that the working condition of the equipment follows a Markov processes with the following transition matrix :*

If it costs Rs. 125 to overhaul a machine (including lost time), on the average, and Rs. 75 in production is lost if a machine is found inoperative. Using steady state probabilities, compute the expected per day cost of maintenance.

Solution. The given transition matrix P can be interpreted as indicating 1/4 of the time it is in fair condition after a day's time, and 3/4 of the time just overhauled machine is in good condition after a day's use. But, a machine which is in good condition has equal chances of still being in good condition or of being in fair condition after a day's use; while a machine in fair condition has equal chances of being in fair or inoperative condition after a day's use. An inoperative machine will be overhauled the next day, so that at the end of the day it would have been just overhauled.

$$P = \textbf{Today}\;\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \textbf{Tomorrow} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Since the given matrix P is an *ergodic regular Markov process,* it will certainly reach to steady state equilibrium. Let the steady state probabilities p_1, p_2, p_3, and p_4 represent the proportion of times that the machine will be in states 1, 2, 3, and 4 respectively.

Now with the help of steady state equations $R = RP$, we have

$$(p_1\ p_2\ p_3\ p_4) = (p_1\ p_2\ p_3\ p_4)\begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In order to find p_1, p_2, p_3 and p_4, we require to solve the simultaneous equations :

$$p_1 = p_4,\ p_2 = 3/4\, p_1 + 1/2\, p_2,\ p_3 = 1/4\, p_1 + 1/2\, p_2 + 1/2\, p_3,\ p_4 = 1/2\, p_3 \quad \text{and} \quad p_1 + p_2 + p_3 + p_4 = 1.$$

Solving these equations, we get $p_1 = 2/11, p_2 = 3/11, p_3 = 4/11$, and $p_4 = 2/11$.

Thus, on an average, 2 out of every 11 days the machine will be *overhauled;* 3 out of every 11 days it will be in *good condition*; 4 out of every 11days it will be in *fair condition;* and 2 out of every 11days it will be found *inoperative* at the end of the days.

Hence the expected (average) cost per day of maintenance will be given by $(2/11)\ 125 + (2/11)\ 75$ = Rs.36.36.

16.10. MARKOV ANALYSIS

In order to explain the Markov analysis, we present here an example of *Brand Switching Models* which emphasises on the time behaviour of customers who make repeated purchases of a product class, but from time to time may switch over from one brand to another. The basic element of a Markov process has to do with *various states.* In *brand switching models,* the state is generally the customer's preference for a particular brand.

Brand Switching Example. Let us consider a consumer sample distributed over two brands A and B, the samples being the representative of the entire group from the standpoint of their brand loyalty and their switching patterns. The behaviour of the large groups can be better described in probabilistic terms. This probabilistic description can be represented by **transition matrix** as explained by the following diagram

Transition Matrix

		To A	To B
From	A	p_{AA}	p_{AB}
	B	p_{BA}	p_{BB}

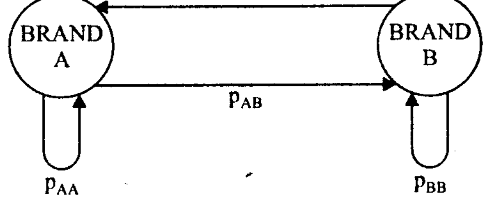


Fig. 16.4 Brand Switching Diagram

In general, for n brands $A_1, A_2, \ldots A_n$, the transition matrix can be represented as follows :

where the probability p_{ij} is such that a customer's preference will switch from *brand i to brand j* from one period to the next.

The most important characteristic of a transition matrix is that

$$\sum_{j=1}^{n} p_{ij} = 1;\ i = 1, 2, ..., n$$

meaning thereby a customer must have some preference.

$$\begin{array}{cc|cccccc|} & & & & \text{To} & & & \\ & & A_1 & A_2 & \cdots & A_j & \cdots & A_n \\ \hline & A_1 & p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1n} \\ & A_2 & p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2n} \\ & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \text{From} & A_i & p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{in} \\ & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ & A_n & p_{n1} & p_{n2} & \cdots & p_{nj} & \cdots & p_{nn} \\ \hline \end{array}$$

16.10-1. Illustrative Examples

Example 5. *Suppose there are two market products of brand A and B, respectively. Let each of these two brands have exactly 50% of the total market in same period and let the market be of a fixed size. The transition matrix is given below :*

$$\begin{array}{cc|cc|} & & \text{To} & \\ & & A & B \\ \hline \text{From} & A & 0.9 & 0.1 \\ & B & 0.5 & 0.5 \\ \hline \end{array}$$

If the initial market share breakdown is 50% for each brand, then determine their market shares in the steady state.

Solution. Here, it is given that the initial state for A and B are 50% each. Then after the promotional efforts made to brands A and B, the transition matrix shows that during second period brand A will retain 90% of its customers and take away 50% of B's so the market share for brand A during the second period will be given by

$$(50\%)\,(0{\cdot}9) + (50\%)\,(0{\cdot}5) = 70\%$$

The corresponding market share for B during the second period will be (50%) (0.1) + (50%)(0.5) = 30%

In matrix form, it can be expressed as $(50\% \quad 50\%)\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (70\% \quad 30\%)$

If tne same transition matrix holds from one period to the other, then market share of two brands for different periods will be as follows :

Period	Brand *A* (Market Share)	Brand *B* (Market Share)
0	50%	50%
1	70%	30%
2	78%	22%
3	81.2%	18.8%
4	82.48%	17·52%
5	82.992%	17.008%
6	83%	17 %

From this table, we observe that starting with 50%, 50% of the market shares, after 6 time periods the resulting market shares are approximately 83% and 17% respectively. So the equilibrium position of market share of A and B will be 5/6 and 1/6 of the total market respectively.

Steady state (or equilibrium) position can be obtained by the matrix equation :

$$(x\ \ y)\begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} = (x\ \ y) \quad \text{or} \quad \begin{cases} 0.9\,x + 0.5\,y = x \\ 0.1\,x + 0.5\,y = y \end{cases} \quad \text{and} \quad x + y = 1.$$

The most important aspect of the steady state is that—if the transition matrix is same throughout, then it is independent of the initial market shares.

Example 6. *Suppose there are three dairies in a town, say A, B and C. They supply all the milk consumed in the town. It is known by all the dairies that consumers switch from dairy to dairy overtime because of advertising, dissatisfaction with service and other reasons. All these dairies maintain records of the number of their customers and the dairy from which they obtained each new customer. Following table illustrates the flow of customers over an observation period of one month, say June.*

Dairy	June 1 (Customers)	Gains from			Losses to			July 1 (Customers)
		A	B	C	A	B	C	
A	200	0	35	25	0	20	20	220
B	500	20	0	20	35	0	15	490
C	300	20	15	0	25	20	0	290

We assume that the matrix of transition probabilites remain fairy stable and that the July market shares are

$$A = 22\%, B = 49\%, \text{ and } C = 29\%.$$

Managers of these dairies are willing to know :

(i) market share of their dairies on Ist August and Ist September,

(ii) their market shares in steady state.

Solution. From the table of the problem, the matrix of the transition probabilities can be easily obtained as follows :

	A	B	C
A	$\frac{160}{200} = 0.80$	$\frac{20}{200} = 0.10$	$\frac{20}{200} = 0.10$
B	$\frac{35}{500} = 0.07$	$\frac{450}{500} = 0.90$	$\frac{15}{500} = 0.03$
C	$\frac{25}{300} = 0.083$	$\frac{20}{300} = 0.067$	$\frac{255}{300} = 0.85$

Market share of the dairies on the *Ist August* will be

$$(0.22 \quad 0.49 \quad 0.29)\begin{pmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{pmatrix} = (0.234 \quad 0.483 \quad 0.283)$$

Market share of the three dairies on Ist september will be

$$(0.234 \quad 0.483 \quad 0.283)\begin{pmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{pmatrix} = (0.245 \quad 0.477 \quad 0.278)$$

The steady state market shares are given by

$$(x \quad y \quad z)\begin{pmatrix} 0.800 & 0.100 & 0.100 \\ 0.070 & 0.900 & 0.030 \\ 0.083 & 0.067 & 0.850 \end{pmatrix} = (x \quad y \quad z) \quad \text{or} \quad \begin{Bmatrix} 0.800x + 0.070y + 0.083z = x \\ 0.100x + 0.900y + 0.067z = y \\ 0.100 + 0.030y + 0.850z = z \\ x + y + z = 1 \end{Bmatrix}$$

and solving these four equations, we get $x = 0.273$, $y = 0.454$, $z = 0.273$.

EXAMINATION PROBEMS

1. A house-wife buys three kinds of cereals: *A, B, C*. She never buys the same cereal on successive weeks. If she buys cereal *A*, then the next week she buys cereal *B*. However, if she buys *B* or *C*, then next week she is three times as likely to buy *A* as the other brand.Find the transition matrix. In the long run, how often she buys each of three brands ?
2. A salesman's territory consists of three cities. *A*, *B* and *C*. He never sells in the same city on successive days. If he sells in city *A*, then the next day he sells in city *B*. However, if he sells in either B or *C*, then the next day he is twice as likely to sell in city *A* as in the other city. In the long run how often does he sell in each of the cities ?
3. On January 1 (this year), bakery *A* had 40 per cent of its local market while the other two bakeries *B* and *C* had 40 per cent and 20 per cent respectively of the market. Based upon a study by a marketing research firm, the following facts were compiled. Bakery *A* retains 90 per cent of its customers while gaining 5 per cent of competitor *B's* customers. Bakery B retains 5 per cent of A's customers and 7 per cent of C 's customers. Bakery *C* retains 83 per cent of its customers and gains 5 per cent of *A*'s customers and 10 per cent of *B* 's customers. What will each firm's share be on January 1, next years and what will each firm's market share be at equilibrium ?
4. Assume that man's profession cas be classified as professional skilled labourer, or unskilled labourer. Assume that the 80 per cent sons of professional men are professionals, 10 per cent are skilled labourers and 10 per cent are unsikilled labourers. In the case of sons of skilled labourers, 60 per cent are skilled labourers, 20 per cent are professionals, and 20 per cent are unskilled labourers. Finally, in the case of unskilled labourers, 50 per cent of the sons are unskilled labourers, 25 per cent each are in the other two categories. Assume that every man has a son and forms a *Markov* chain by following a given family through several generations. Set-up the matrix of transition probabilities. Find the probability that grandson of an unskilled labourer is a proferssional man.
5. *Honey* Inc. had 35% of the local market for its cosmetics, while the two other manufacturers of cosmetics *Lace Inc.* and *Shalon* Inc. have 40% and 25% shares respectively in the local market, as on Ist April of this year. A study by a market research firm has disclosed the following.
   Honey Inc. retains 86% of its customers, while it gains 4% and 6% of the customers from its two competitors, *Lace* and *Shalon* respectively. *Lace* Inc. retains 90% of its customers, and gains 8% and 9% of customers respectively from *Heney* and *Shalon*. Shalon retains 85% of its customers and gains 6% and 6% of customers from Lace and Honey respectively.
   What will be the share of each firm on April I next year ? What will be the market share of each firm at equilibrium ?
   [**Ans.** *Honey* Inc. 33.2%, *Lace* Inc. 41.05% and *Shalon Inc.* 25.75%, Market share of each firm at equilibrium will be 25%, 46% and 29%]

Example 7. *The number of units of an item that are withdrawn from inventroy on a day-to-day basis is a Markov chain process in which requirements for tomorrow depend on today's requirements. A one day transition matrix is given below :*

Number of units withdrown from inventory

Tomorrow

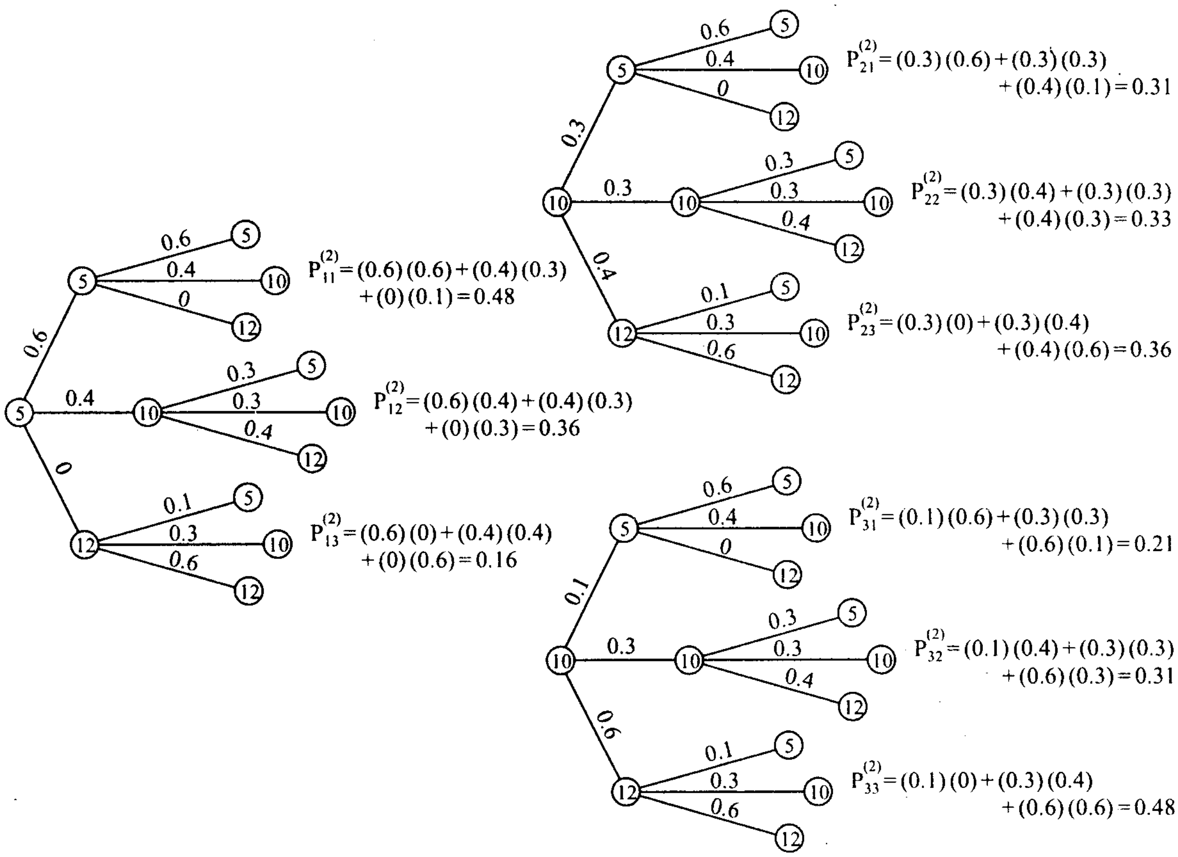
$$\text{Today}\quad \begin{array}{c} \\ 5 \\ 10 \\ 12 \end{array}\begin{array}{c} \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \end{array}$$

(i) Construct a tree diagram showing inventory requirements on two consecutive days.

(ii) Develop a two-day transition matrix.

(iii) Comment on "how a two day transition matrix might be helpful to a manager who is responsible for inventory management."

Solution. *(i)*



(ii) If the transition matrix be denoted by P, the two-day transition matrix is given by

$$P^2 = \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} = \begin{array}{c} \\ 5 \\ 10 \\ 12 \end{array}\begin{array}{c} \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \begin{bmatrix} 0.48 & 0.36 & 0.16 \\ 0.31 & 0.33 & 0.36 \\ 0.21 & 0.31 & 0.48 \end{bmatrix} \end{array}$$

(iii) Suppose that every morning a manager must place an order for inventory replenishment. Because of delivery time requirements, an order placed today arrives 2 days later. The two day transition matrix can be used for

guiding ordering decisions. For example, if today the manager experiences a demand for five units, then two days later (when replenishment stock arrives in response to today's order) the probability of requiring 5 units is 0.48, 10 units is 0.36 and that of 12 is 0.16.

Q. 1. What do you understand by Markov chains ? Expain how it can be used for predicting sales-force needs.

2. What do you understand by Markov processes ? In what areas of management can it be applied successfully. **[Delhi MBA(PT) Nov. 95]**

3. Explain how Markov processes can be used by a company to predict its manpower needs.

4. Explain the terms : (i) Markov process, (ii) Transition probabilities, (iii) Matrix of transition probabilities, (iv) Ergodic process, (v) Equilibrium of steady state.

5. What are the three fundamental properties of a finite state, first order Markov process.

6. Explain how decision tree helps to understand the problem of Markov processes.

SELF EXAMINATION PROBLEMS

1. A grocer stocks his store with three types of detergents A, B and C. When brand A is sold out the probability is 0.7 that he stocks up with brand A again. When he sells out brand B, the probability is 0.8 that he will stock up again with brand B. Finally when he sells out brand C, the probability is 0.6 that he will stock up with brand C again. When he switches to another detergent he does so with equal probability for the remaining two brands. Find the transition matrix. In the long run how does he stock up with detergents ?
[**Ans.** $A = 0.3077$, $B = 0.4615$, $C = 0.2308$]

2. Consider a certain community in well-defined area with three types of grocery stores; for simplicity we shall call them I, II and III, Within this community (we assume that the population is fixed) there always exists a shift of customers from one grocery store to another. A study was made on January 1 and it was found that 1/4 shopped at store I, 1/3 at store II and 5/12 at store III. Each month store I retains 90 per cent of its customers and loses 10 per cent of them to store II. Store II retains 90 per cent of its customers and loses 85 per cent of them to store I and 10 per cent of the to store III. Store III retains 40 per cent of its customers and loses 50 per cent of them to store I and 10 per cent to store II.
(i) What proportion of customers will each store retain by February 1 : March 1 ?
(ii) Assuming the same pattern continues, what will be the long-run distribution of customers among the three stores?
[**Ans** (*i*) Feb.1 : I= 0.7166, II = 0.0832, III = 0.1999; March 1 : I = 0.8155, II= 0.0956, III = 0.0882;
(*ii*) I = 0.8888, II = 0.0952, III 0.0158]

3. A salesman territory consists of cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However, if he sells in either B or C, then the next day he is twice as likely to sell in city A as in the other city. In the long run, how often does he sell in each of the cities ?
[**Ans** city A = 40%, city B = 45%, city C = 15%]

4. Consider the following transition matrix for brand switching :
The manufacturer of brand C are considering a marketing strategy to attract brand B customers. It is estimated that this strategy will :
(a) increase probability of customer's switch from B to C by 0.20,
(b) decrease probability of customer's switch from C to B by 0.10, and
(c) decrease probability of customer's switch from A to C by 0.25.
Should the new strategy be used ?

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.50 & 0.15 & 0.35 \\ 0.20 & 0.55 & 0.25 \\ 0.25 & 0.30 & 0.45 \end{bmatrix} \end{array}$$

5. The XYZ Company is planning an extensive advertising campaign to increase the company's market share. The company is faced with the job of choosing between the two campaings that have been recommended. It has decided to test each proposal in two test areas where the initial market shares of the competing firms and the initial transition probability matrices are the same. Also the market shares of the firms are close to their national average, which are : brand X (of XYZ Co), 28 per cent ; brand A, 39 per cent; and brand B, 33 per cent. In the two test areas, the market shares are : brand X, 30 per cent; brand A, 40 per cent; and brand B, 30 per cent. The matrix of initial transition probabilities for both areas is :

$$\begin{array}{c} \\ \\ \text{Brand } X \\ \text{Brand } A \\ \text{Brand } B \end{array}\begin{array}{c} \begin{array}{ccc} \text{Brand} & \text{Brand} & \text{Brand} \\ X & A & B \end{array} \\ \begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.7 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \end{array}$$

At the finish of two different advertising problems in two test areas, the transition probabilities which were determined, are :

Test Area 1

$$\begin{array}{c} \\ \\ \text{Brand } X \\ \text{Brand } A \\ \text{Brand } B \end{array}\begin{array}{c} \begin{array}{ccc} \text{Brand} & \text{Brand} & \text{Brand} \\ X & A & B \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.8 \end{bmatrix} \end{array}$$

Test Area 2

$$\begin{array}{c} \\ \\ \text{Brand } X \\ \text{Brand } A \\ \text{Brand } B \end{array}\begin{array}{c} \begin{array}{ccc} \text{Brand} & \text{Brand} & \text{Brand} \\ X & A & B \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} \end{array}$$

Assuming the advertising campaigns are equal in terms of cost, which advertising campaign gives the highest market share at equilibrium ?

6. A market survery is made on three brands of breakfast foods X, Y and Z. Every time the customer purchases a new package, he may buy the same brand or switch to another brand. The following estimates are obained, expressed as decimal fraction :

$$\begin{array}{ll} & \textbf{Brand Just Purchased} \\ & \begin{array}{ccc} X & Y & Z \end{array} \\ \textbf{Present Brand} \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

At this time it is estimated that 30 per cent of the people buy brand X, 20 per cent brand Y and 50 per cent brand Z. What will the distribution of customers be two time periods later and at equilibrium ?

7. A professor has three pet questions, one of which occurs on every test he gives. The students know his habits well. He never uses the same question twice in a row. If he used question one last time, he tosses a coin and uses question two if a head comes up. If he used question two, he tosses two coins and switches to question three if both come up heads. If he used question three, he tosses three coins and switches to question one if all three come up heads.In the long run which question does he use most often and with how much frequency is it used ?
[**Ans.** Question two = 40%]

8. (a) The purchase patterns of two brands of toothpaste can be expressed as a Markov process with the following transition probabilities :

$$\begin{array}{ll} & \begin{array}{cc} \text{Formula } X & \text{Formula } Y \end{array} \\ \begin{array}{r} \text{Formula} X \\ \text{Formula} Y \end{array} & \begin{bmatrix} 0.90 & 0.10 \\ 0.05 & 0.95 \end{bmatrix} \end{array}$$

(*i*) Which brand appears to have most loyal customers ? Explain.

(*ii*) What are the projected market shares for the two brands ?

(b) Suppose that in problem (a) a new toothpaste brand enters the market such that the following transition probabilities exist :

$$\begin{array}{ll} & \begin{array}{ccc} \text{Formula } X & \text{Formula } Y & \text{Formula } Z \end{array} \\ \begin{array}{r} \text{Formula} X \\ \text{Formula} Y \\ \text{Formula} Z \end{array} & \begin{bmatrix} 0.80 & 0.10 & 0.10 \\ 0.05 & 0.75 & 0.20 \\ 0.40 & 0.30 & 0.30 \end{bmatrix} \end{array}$$

What are the new long-run market shares ?

Which brand will suffer most from the introduction of the new brand of toothpaste ?

9. Bajaj manufactures and sells the *Chetak* scooters. The two closest competitors market one brand called the ***Vijay*** and another brand called the *Rajdoot*. Because of the custom manufacturing processes and their inherent high costs, no other competitor has any effect on the current market. The year 1986 was an exceptionally good year in terms of gain lose trade offs. The year's activity is summarized in the following table.

Brand	Period One (Customers)	Changes During Period		Period Two (Customers)
		Gain	Loss	
Chetak (C)	500	30	25	505
Vijay (V)	500	20	10	510
Rajdoot (R)	500	10	25	485

Further analysis resulted in the gain-loss summary below :

Brand	Period Ond (Customers)	Gains From			Losses To			Period Two (Customers)
		C	V	R	C	V	R	
Chetak (C)	500	0	10	20	0	15	10	505
Vijay (V)	500	15	0	5	10	0	0	510
Rajdoot (R)	500	10	0	0	20	5	0	485

Bjaj's mangement wants to know the current rate of gains and losses. In addition, they want to kown the expected future market share of each firm over a three year period and the state at which market equilibrium will exist (if it can).

10. A Professor tried not to be late for class too often. If he is late one day, he is 90 per cent sure to be on time next time. If he is on time, then the next time there is a 30 per cent chance of his being late. In the long run, how often is he late for class ?

11. (a) Given the following transition matrix determine the equilibrium market share for each company.

$$\begin{array}{c} \\ \text{Co.}X \\ \text{Co.}Y \\ \text{Co.}Z \end{array}\begin{array}{c} \begin{array}{ccc} \text{Co.}X & \text{Co.}Y & \text{Co.}Z \end{array} \\ \begin{bmatrix} 0.25 & 0.05 & 0.01 \\ 0.25 & 0.85 & 0.20 \\ 0.50 & 0.10 & 0.70 \end{bmatrix} \end{array}$$

(b) In an effort to improve its market share, company X considers two marketing strategies. The first is to provide a coupon with its products that the customers can apply to his or her next purchase. Data indicate that this strategy affects the loyalty of company A's own customers as reflected in the following transition matrix :

$$\begin{array}{c} \\ \text{Co.}X \\ \text{Co.}Y \\ \text{Co.}Z \end{array}\begin{array}{c} \begin{array}{ccc} \text{Co.}X & \text{Co.}Y & \text{Co.}Z \end{array} \\ \begin{bmatrix} 0.50 & 0.05 & 0.10 \\ 0.10 & 0.85 & 0.20 \\ 0.40 & 0.10 & 0.70 \end{bmatrix} \end{array}$$

Instead of improving retention probabilities, the second strategy is designed to gain customers from company Y and Z.The transition matrix resulting from the second market strategy is as follows :

$$\begin{array}{c} \\ \text{Co.}X \\ \text{Co.}Y \\ \text{Co.}Z \end{array}\begin{array}{c} \begin{array}{ccc} \text{Co.}X & \text{Co.}Y & \text{Co.}Z \end{array} \\ \begin{bmatrix} 0.25 & 0.10 & 0.20 \\ 0.25 & 0.85 & 0.10 \\ 0.50 & 0.05 & 0.70 \end{bmatrix} \end{array}$$

Suppose that the costs for both market strategies are the same. If you were the market analyst for company X, which market strategy would you recommed ? Why ?

[Ans. (a) $X = 00833$, $Y = 0.5833$, $Z = 0.3333$, (b) $X = 0.1584$, $Y = 0.4950$, $Z = 0.3466$]

12. The School of International Studies for Population found out by its survey that the mobility of the population of a state to the village, town and city is in the following percentage :

		To		
		Village	*Town*	*City*
From	*Village*	50%	39%	20%
	Town	10%	70%	20%
	City	10%	40%	50%

What will be the proportion of population in vilage, town and city after two years, given that the present population has proportion of 0.7, 0.2 and 0.1 in the village, town and city respectively ? What will be the respective proportions in the long run ?

[Ans. (i) $X = 0.225$, $Y = 0.479$, $Z = 0.269$, (ii) $X = 0.16667$, $Y = 0.5476$, $Z = 0.2857$]

13. Suppose that new razor blades were introduced, in the market by three companies at the same time. When they were introduced, each company had an equal share of the market, but during the first year the following changes took place :

(*i*) Company A retained 90 per cent of its customers, lost 3 per cent to B, and 7 per cent to C.

(*ii*) Company B retained 70 per cent of its customers, lost 10 per cent to A, and 20 per cent to C.

(*iii*) Company C retained 80 per cent of its customers, lost 10 per cent to A, and 10 per cent to B.

Assuming that no changes in the buying habits of the consumer occur,

(a) what are the market shares of the three companies at the end of the first year ? The second year ?

(b) what are the long run market shares of the three companies ?

OBJECTIVE QUESTIONS

1. Probabilities of occurrence of any state are
(a) collectively exhaustive.
(b) mutually exclusive.
(c) representing one of the finite number of states of nature in the system.
(d) all of the above.

2. In a matrix of transition probability, the probability values should add up to one in each
(a) row. (b) column. (c) diagonal. (d) all of the above.

3. In matrix of transition probability, the element a_{ij} where $i = j$ is a
(a) gain. (b) loss. (c) retention. (d) none of the above.

4. In Markov analysis, state probabilities must
(a) sum to one. (b) be less than one.
(c) be greater than one. (d) none of the above.

5. If a matrix of transition probability is of the order $n \times n$, then the number of equilibrium equations would be
(a) n. (b) $n - 1$. (c) $n + 1$. (d) none of the above.

6. In the long run, the state probabilities become between 0 and 1
(a) in no case. (b) in some cases. (c) in all cases. (d) cannot say.

7. While calculating equilibrium probabilities for a Markov process, its assumed that
   (a) there is a single absorbing state.
   (b) transition probabilities do not change.
   (c) there is a single non-absorbing state.
   (d) none of the above.
8. The first order Markov chain is generally used when:
   (a) transition probabilities are fairly stable.
   (b) change in transition probabilities is random.
   (c) no sufficient data are available.
   (d) all of the above.
9. Which of the following is not one of the assumptions of Markov analysis ?
   (a) There are a limited number of possible states.
   (b) A future state can be predicted from the preceding one.
   (c) There are limited number of future periods.
   (d) All of the above.
10. Markov analysis is useful to answer following questions
    (a) predicting the state of the system at some future time?
    (b) calculating transition probabilities at some future time?
    (c) all of the above.
    (d) none of the above.

Answers

1. (d) **2.** (a) **3.** (c) **4.** (a) **5.** (c) **6.** (c) **7.** (b) **8.** (a) **9.** (c) **10.** (c).

✸✸✸

17

SIMULATION
(Monte-Carlo Technique)

17.1. INTRODUCTION

It is evident that there are many problems of real life which cannot be represented mathematically due to the stochastic nature of the problem, the complexity in problem formulation, or the conflicting ideas needed to properly describe the problem under study. Under such circumstances *simulation* is often used when all else fail. This method is often viewed as a *"method of last resort."*

Simulation analysis is a natural and logical extension to the analytical and mathematical techniques used for solving the problems in Operations Research. Simulation which can appropriately be known as *management laboratory*, determines the effect of alternate policies without disturbing the real system. Recent advances in simulation methodologies, software availability, and technical developments have made simulation one of the most widely used and popularly accepted tool in *'operations research'* and *'systems analysis'*. It helps us in deciding the best policy with the prior assurances that its implementation will certainly prove to be beneficial to the organization.

The simulation technique has long been applied by the analysts and designers in physical sciences and it has now become an important tool for dealing with the complicated problems of managerial decision making. The simulated models of aircrafts are tested in wind tunnels to examine their aerodynamic characteristics and the scale models of machines are used to simulate the plant layouts.

The first important application of simulation was probably made by ***John Von Neumann*** and ***Stanislaw Ulam*** for determining the complicated behaviour of neutrons in a nuclear shielding problem being too complex for mathematical analysis. After getting the remarkable success of this technique on neutron problem, it became more popular and found many applications in business and industry. In early 1950s, the development of digital computer further increased the rapid progress in the simulation techniques.

Simulation is one of the easiest tools of management science to use, but probably one of the hardest to apply properly and perhaps most difficult from which to draw accurate conclusions. Due to widespread availability of digital computers, simulation becomes readily available to most of the engineers and managers engaged in operations research projects. Regardless of these drawbacks, simulation is a useful technique and one which is specially suitable for complicated *operations research* and *systems analysis* problems.

The purpose of this chapter is to examine the process of simulation and necessary tools to perform such analysis. A special emphasis is given to the *Monte-Carlo method* of simulation. Some simple examples are discussed to explain the *Monte-Carlo technique.* To obtain the reliable results the use of computer is very essential in all simulation problems. But for easy demonstration of simulation technique, the numerical examples have been solved by hand computations only.

17.2 WHAT IS SIMULATION ?

In fact, simulation is the representative model for real situations. While visiting some trade-fairs and exhibitions we often find a number of simulated environments therein. For example, a children's cycling park with various signals and crossings in the exhibition is a simulated (represented) model of city-traffic in real system. Also, a simple example is the testing of an aircraft model in a wind tunnel from which the performance of the real aircraft is determined for being fit under real operating conditions. In the laboratories we often perform a number of experiments on simulated models to predict the behaviour of the real system under true environments. The environments in a *museum of natural history* and in a *geological garden* are also good examples of simulation.

Another idea of simulation is involved inflight simulators for training pilots. A computer directs the student's handling of the controls in a simulated aeroplane flight deck. The instruments are then operated by the computer to give the same readings which they would in a real flight. An instructor can intervene with 'catastrophes' like an engine failure or a bad strom and a television camera is moved over a model of some countryside to give the trainee visual feedback of how the aircraft is behaving.

The combination of computing and simulation has also resulted in the production of *TV games.* Players interrupt the way a computer program moves various images around the screen from a keyboard or hand-held controller. The computer incorporates their responses into these movements in accordance with the rules of the particular game. Incidentally, such programs make extensive use of random numbers to find the deflection of tennis balls, the positioning of hostile space ships, etc.

Actually the idea of simulating real system for enjoyment purposes is already known to us. The *chess-playing game* is a non-probabilistic simulation of a fight between the *black* and *white* armies. The game of *snake and ladders* was initially proposed to simulate the moral progress of the players who moved up ladders when they were 'good' and fell down snakes, indicating temptation, when they were bad. Like in many other board games, dice are used as random number generators.

In all these examples, we have tried to represent the reality to observe — what would happen under real operating situations. Thus, such representation of reality, which may be either in physical form or in a mathematical equations form, may be called ***simulation.***

Q. What is similuation technique ? **[Meerut (M.Com.) 2005]**

17.3. DEFINITIONS OF SIMULATION

Before we proceed further, it becomes necessary to define the term simulation in more sutiable forms. Following few definitions are given below :

Definition 1. *Simulation is a representation of reality through the use of a model or other device which will react in the same manner as reality under a given set of conditions.*

Definition 2. *Simulation is the use of system model that has designed the characteristics of reality in order to produce the essence of actual operation.*

Definition 3. *According to* ***Donald G. Malcolm****, a simulated model may be defined as one which depicts the working of a large scale system of men, machines, materials and information operating over a period of time in a simulated environment of the actual real world conditions.*

Definition 4. *According to* ***T.H. Naylor*** *et al.* (1966), *simulation is a numerical techinque for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and structure of a complex real-world system over extended periods of time.*

Definition 5. ***Churchman*** *has defined simulation as follows* :

"X simulates Y" is true if and only if : (a) *X and Y are formal systems;* (b) *Y is taken to be the real system;* (c) *X is taken to be an approximation to the real system; and* (d) *the rules of validity in X are non-error-free, otherwise X will become the real system.*

17.4. TYPES OF SIMULATION

Simulation is mainly of two types :

(*i*) **Analogue Simulation (or Environmental Simulation).** The simple examples cited is **Sec. 17.2.** are of simulating the reality in physical form, which we may refer as ***analogue (or environmental) simulation***

(*ii*) **Computer Simulation (or System Simulation).** For the complex and intricate problems of managerial decison making, the *analogue simulation may* not be applicable, and the actual experimentation with the system may be uneconomical also. Under these situations, the complex system is formulated into a mathematical model for which a computer programme is developed, and then the problem is solved by using high speed electronic computer. Such type of simulation is called a ***computer simulation or system simulation.***

The simulation models can be classified into following **four** categories :

(a) Deterministic models. In these models, *input* and *output* variables are not permitted to be random variables and models are described by *exact functional relationship.*

(b) Stochastic models. In these models, at least one of the variables or functional relationship is given by *probability functions.*

(c) Static models. These models do not take variable time into consideration.

(d) Dynamic models. These models deal with time varying interaction.

17.5 WHY SIMULATION IS USED ?

It has been already discussed in the chapter on **"What is Operations Research ?"** that mainly following techniques are adopted for solving various types of managerial decision making problems in *operations research.*

(*i*) *Scientific method,* (*ii*) *Analytical method,* and (*iii*) *Iterative method.* But each method has its own drawbacks and limitations as discussed below :

1. Drawbacks of Scientific Method. The steps of scientific method have the following limitations and difficulties :

(*i*) It may be either impossible or extremely costly to observe certain processes in the real life situations.

(*ii*) The observed system may be so complex that it may be impossible to describe it in terms of a set of mathematical equations.

(*iii*) Even though the mathematical model can be formulated to describe system under study, a straightforward analytical solutions may not be available. For example, such situations may arise in *complex queueing problems, job-shop problems, high order difference equations, complicated stochastic models, multi-integral problems.* etc.

(*iv*) It may be either impossible or very costly to perform validating experiments on mathematical models describing the system.

Thus on account of these drawbacks the scientific method cannot be used to solve complex managerial decison-making problems.

2. Drawbacks of Analytical Method. Analytical techniques used in *dynamic programming, queueing theory, network models,* etc.,are not sufficient to tackle all the important managerial problems requiring data analysis due to following limitations :

(*i*) *Dynamic programming models* (discussed in **Unit 5** of this book), however, can be used to determine optimal strategies taking into account the uncertainties and can analyse multiperiod planning problems. But, still it has its own shortcomings. Dynamic programming models can be used to tackle very simple situations involving a very few variables. If the number of state variables becomes larger, the computation work becomes quite complex and difficult.

(*ii*) Similar limitations also hold good for other mathematical techniques like *dynamic stochastic models* such as *inventory* and *waiting line situations.* Only small scale systems are amenable to these models. But by making a number of assumptions the systems are simplified to such an extent that in many cases the results thus obtained are only rough approximations.

3. Drawbacks of Iterative Method.

(*i*) In *linear programming models,* we assume that data does not change over the entire planning horizon. This is one time decision process and assumes average values for the decision variables. If the planning horizon is long, say 15 years, the *multi-period linear programming* model may deal with the yearly averaged data, but will not take into account the variations over the months and weeks. Consequently, month to month and week to week operations are left implicit.

(*ii*) Other important limitation of linear programming is that it assumes that data should be known with certainty. In many real situations, the uncertainties about the data are such that they cannot be ignored. In case the uncertainty relates to only a few variables, the *sensitivity analysis* can be used to dtermine its effect on the decision. But, in the situations, where uncertainty pervades the entire model, the *sensitivity analysis* may become too cumbersome and computationally difficult to determine the impact of uncertainty on recommended plan.

From above mentioned drawbacks, we conclude that whenever the characteristics like ***uncertainty***, *complexity, dynamic interaction* between the *decision* and *subsequent event* and the need to develop detailed procedures and finely divided time intervals, all combined together in one situation, then model becomes too complex to be solved by any of the techniques of mathematical programming and probabilistic models. Then such complex model must be analysed by some other kind of quantitative technique, which may give quite accurate and reliable results. Many new techniques are investigated so far, but among all, the best available is *'simulation'*.

In general, the simulation technique is a dependable tool in situations where mathematical analysis is either too complex or too expensive.

Why 'simulation' is used for solving real-life problems ?

There are four main reasons for using simulation for solving real-life problems as mentioned below :

1. *Simulation techniques allow experimentation with a model of the real-life system rather than the actual operating system.*
   Sometimes experimenting with the actual system itself could prove to be too expensive and in several cases too disruptive. For example, if we compare two different ways of providing food service in a hospital, the confusion that may arise from operation of two different systems long enough to get valid observations might be too great. Similarly, the operation of a large computer centre under a number of different *operating* alternative might be too costly to be feasible.
2. *Sometimes there is no sufficient time to allow the actual system to operate extensively :*
   For example, if we want to study long term trends in world population, it is not possible to wait for desired number of years to see the results. Simulation allows to incorporate time into an analysis. In a computer situation of business operation the manager can compress the result of several years or periods into a few minutes of running time.
3. *The non-technical manager can comprehend simulation more easily than a complex mathematical model :*
   Simulation does not require simplifications and assumptions to the extend needed in analytical solutions. A simulation model is easier to explain to management personnel since it is a description of behaviour of some system or process.
4. *The use of simulation enables a manager to provide insights into certain managerial problems* where analytical solutions of a model is not possible or where the actual environment is difficult to observe. For example, simulation is used in space flights or the charting satellite.

17.6. LIMITATIONS OF SIMULATION TECHNIQUE

Although many operations research analysts consider the simulation as a *method of last resort* and use it only when all other techniques fail. If the problem can be well represented by a mathematical model, the analytical method is considered to be more economical, accurate and reliable. But in the case of very large and complex problems simulation may suffer the similar drawbacks as other mathematical models. The limitations of the simulation technique can be briefly outlined as below :

(*i*) Optimum results cannot be produced by *simulation.* Since the model mostly deals with uncertainties, the results of simulation are only reliable approximations involving statistical errors.

(*ii*) The another difficulty lies in the quantification of the variables. In many situations, it is not possible to quantify all the variables which affect the behaviour of the system.

(*iii*) In very large and complex problems, it becomes difficult to make the computer program on account of large number of variables and the involved inter-relationships among them. The number of variables may be too large and may exceed beyond the capacity of the available computer.

(*iv*) For the problems requiring use of computer, the simulation is by no means a less costly method of analysis. Thus, simulation is comparatively costlier and time consuming method in many situations.

(*v*) Other important limitations stem from too much tendency to rely on the simulation models. Consequently, this technique is also applied to some simple problems which can otherwise be solved by more appropriate techniques of mathematical programming.

Q. 1. Explain the Monte-Carlo technique and its limitations
2. What are major limitations of simulation.
3. Mention any four reasons for using simulation for solving management problems. **[C.A. (May) 93]**
4. What do you understand by simulation ? Explain briefly its limitations and advantages too ? **[IGNOU 2001, 2000, 99]**

17.7. PHASES OF SIMULATION MODEL

A simulation model mainly consists of two basic phases :

Phase 1 : Data Generation . Data generation involves the sample observation of variables and can be carried out with the help of any of the following methods :

(*i*) using the random number tables;

(*ii*) resorting to mechanical devices (for example, roulettes wheel); and (*iii*) using electronic computers.

Phase 2 : Book-keeping. The book-keeping phase of a simulation model deals with updating the system when new events occur, monitoring and recording the system states as and when they change; and keeping track of quantities of our interest (such as, *idle time* and *waiting time*) to compute the measures of effectiveness.

17.8. EVENT-TYPE SIMULATION

The ***event type simulation*** can be understood by discussing the following example.

Example 1. *Consider a situation where customers arrive at a one-man barber shop for hair cutting. The problem is to analyse the system in order to evaluate the quality of service and the economic feasibility of offering the service. To measure the quality of service one has to make the assessment of the average waiting time per customer and the percentage of time the barber remains idle. Construct the simulation model.*

Solution : Construction of the model. To construct the model of this system we observe that the changes involved in the analysis of the system can occur only when a customer arrives for service or departs after completion of service. When a customer reaches the barber's shop, he will have to wait, if the server (barber) is busy. But, on the other hand, a departure of customer after the completion of his service (hair cut) indicates that the server is availabe to serve the waiting customers, if any one is present.

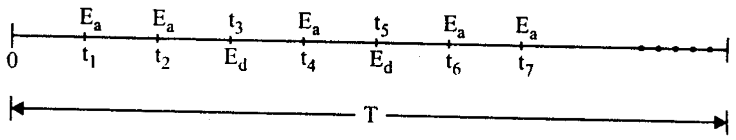


Fig. 17.1

From this, we conclude that only two-types of events can occur, *i.e.* an ***arrival event*** and a ***departure event.*** It shows that as the simulator progresses on the time scale, we must pay due attention to one system whenever an event occurs.

Let us denote the arrival event by E_a, the departure event by E_d, and the simulated period (time span) by T. The simulator starts at time 0 and first progresses up to $t = t_1$, then $t = t_2$ and so on, until the entire simulated period T is covered. The above diagram (Fig. 17.1) shows the occurrences of event E_a and E_d over the time period T, where the simulation starts by generating E_a at t_1. In the beginning, when the service facility if free, the service of the customer will be started immediately. Then the following two new events must be generated :

(*i*) the next arrival may occur, (*ii*) the service of the customer may be completed.

The next arrival can be determined from inter-arrival time. Thus E_a is determined at time t_2. The departure time of the customer in service is obtained from service time, and thus event E_d is generated at time t_3 . Now, both the events E_a (at t_1) and E_d (at t_3) are listed in chronological order, so that the simulator may recognize that the event E_a occurs before E_d. The next event under consideration is E_a at t_2 and at this moment E_a at t_1 is deleted from the stored list (because of past event). The event E_a at t_2 now generates E_a at t_4. Since the service facility is busy, the new arriving customer E_a (at t_4) joins a waiting line. Now, E_a at t_4 is deleted from the list and E_d at t_5 is considered next. At this time a customer is taken from the waiting line and departure event E_d at t_5 is generated. This process is repeated until the entire simulated period T is covered.

17.8-1 Illustrative Examples

Example 2. *Customers arrive at a milk booth for the required service. Assume that inter-arrival and service times are constant and given by 1.8 and 4 time units, respectively. Simulate the system by hand computations for 14 time units. What is the average waiting time per customer ? What is the percentage idle time of the facility ? (Assume that the system starts at $t = 0$).*

Solution. Since the facility is free initially, the service of first customer will be started. So its departure time becomes $t = 0 + 4 = 4$ time units. Next event (arrival) occurs at $t = 0 + 1.8 = 1.8$ time units, which is listed before E_d at $t = 4$. Now the service facility is still busy, second customer stands in the queue and the first one is to be considered in this queue. The third arriving customer (new arrival event) occurs at $t = 1.8 + 1.8 = 3.6$ which precedes E_d at $t = 4$. Again, third customer stands in the queue and a new arrival event E_a (fourth customer) occurs at $t = 3.6 + 1.8 = 5.4$. This event succeeds E_d at $t = 4$. At this moment first customer departs leaving the service facility free. Second customer, who was the first to join the queue, now gets his chance for service. The waiting time is determined as the time period from the moment he joined the queue and his service is started. This process is repeated until the simulated period is completed. The results of simulation are tabulated as below :

(see simulation table)

Simulation Table

Time	Event	Customer	Waiting Time
0.0	E_a	1	–
1.8	E_a	2	–
3.6	E_a	3	–
4.0	E_d	1	4–1.8 = 2.2 (Customer 2)
5.4	E_a	4	–
7.2	E_a	5	–
8.0	E_d	2	8–3.6 = 4.4 (Customer 3)
9.0	E_a	6	–
10.8	E_a	7	–
12.0	E_d	3	12–5.4 = 6.6 (Customer 4)
13.6	E_a	8	–
14.0	End	–	14– 7.2 = 6.8 (Customer 5) 14– 9.0 = 5.0 (Customer 6) 14–10.8 = 3.2 (Customer 7) 14–13.6 = 0.4 (Customer 8)

From this simulation table it is evident that

(*i*) average time per customer,

$$= \frac{2.2 + 4.4 + 6.6 + 6.8 + 5.0 + 3.2 + 0.4}{8}$$

= 3.57 time units,

(*ii*) average waiting time per customer for those who must wait, $\frac{28.6}{7} = 4.08$,

(*iii*) percentage idle time of the service facility = 0%

Example 3. *A town contains six wards and they contain 170, 510, 640, 75, 250 and 960 houses respectively. Make a random selection of 8 houses using the tables of random numbers. Explain the procedure adopted by you.*

Solution :

Ward number (1)	No. of houses (2)	Cum. no. of houses (3)	Random nos. fitted against the ward as per column (3) (4)
1	170	170	0136, 0163
2	510	680	0472
3	640	1320	1155, 1089, 1221
4	75	1395	
5	250	1645	
6	960	2605	2579, 2570

We enter the random number from the table of *Random Numbers* (refer Table 6-B, page 6.37). The first random number taken from the table is 2579. Since it is immediately ≤ 2605 in column (3) of above table, it is fitted against ward number 6 in column (4) The next random number is 6378 which is greater than 2605 in column (3), therefore, it is dropped. In this manner, the following random numbers are either fitted in column (4) or they are dropped : 6134 dropped, 3142 dropped ; 1155 (fitted against ward number 3; 8972 dropped; 0136 (fitted against ward number 1); 4941 dropped; 8614 dropped; 5309 dropped, 6188 dropped; 1089 fitted against ward number 3; 0163 fitted against ward number 1; 0472 fitted against ward number 2; 1221 fitted against ward number 3; 5922 dropped; 4773 dropped; 2570 fitted against ward number 6.

Since 8 houses have been selected, we stop the process. Thus a random slection of 8 houses is :

Two from ward 1, *one* from ward 2, *three* from ward 3 and *two* from ward 6.

Example 4. *Dr. STRONG is a dentist who schedules all per patients for 30 minutes appointments. Some of the patients take more or less than 30 minutes depending on the type of dental work to be done. The following summary shows the various categories of work, their probabilities and the time actually needed to complete the work.*

Categories	Filling	Crown	Cleaning	Extraction	Checkup
Time required (min.) :	45	60	15	45	15
Prob. of category :	0.40	0.15	0.15	0.10	0.20

Simulate the dentist's clinic for four hours and determine the average waiting time for the patients as well as the illness of the doctor. Assume that all the patients show up at the clinic at exactly their scheduled arrival time starting at 8.00 am. Use the following random numbers for handling the above problem :

40 82 11 34 25 66 17 79 **[C.A. (Nov.) 90]**

Solution. If the numbers 00–99 are allocated in proportion to the probabilities associated with each categories of work, then various kinds of dental work can be sampled, using random number table :

Category	:	Filling	Crown	Cleaning	Extraction	Checkup
Probability	:	0.40	0.15	0.15	0.10	0.20
Cum. Prob.	:	0.40	0.55	0.70	0.80	1.00
Rand. Nos.	:	00–39	40–54	55–69	70–79	80-99

Using the given random numbers, a work sheet can now be completed as follows :

Future Events

Patient	Scheduled Arrival	Randon No.	Category	Service Time (min.)
1	8.00 AM	40	Crown	60
2	8.30 AM	82	Checkup	15
3	9.00 AM	11	Filling	45
4	9.30 AM	34	Filling	45
5	10.00 AM	25	Filling	45
6	10.30 AM	66	Cleaning	15
7	11.00 AM	17	Filling	45
8	11.30 AM	79	Extraction	45

Now let us simulate the dentist's clinic for four hours starting from 8.00 AM.

Status

Time	Event (Arrival/Departure)	Patient No. (Service Time)	Patients Waiting
8.00 AM	Ist arrives	Ist (60 min.)	—
8.30 AM	2nd arrives	Ist (30 min.)	2nd
9.00 AM	Ist departs, 3rd arives	2nd (15 min)	3rd
9.15 AM	2nd departs	3rd (45 min)	—
9.30 AM	4th arrives	3rd (30 min.)	4th
10.00 AM	3rd departs, 5th arrives	4th (45 min.)	5th
10.30 AM	6th arrives	4th (15 min.)	5th & 6th
10.45 AM	4th departs	5th (45 min.)	6th
11.00 AM	7th arrives	5th (30 min.)	6th & 7th

11.30 AM	5th departs, 8th arrives	6th (15 min.)	7th & 8th
11.45 AM	6th departs	7th (45 min.)	8th
12.00 Noon	End	7th (30 min.)	8th
12.30 PM		8th (45 min.)	

This table shows that the dentist was not idle during the entire simulated period.

The waiting times for the patients were as follows :

Patient	:	1	2	3	4	5	6	7	8
Arrival	:	8.00	8.30	9.00	9.30	10.00	10.30	11.00	11.30
Service starts	:	8.00	9.00	9.15	10.00	10.45	11.30	11.45	12.30
Waiting (min.)	:	0	30	15	30	45	60	45	60 Total 2805

The average waiting time of a patient was 285/8=35.625 min.

EXAMINATION PROBLEMS

1. With the help of a single server queueing model having inter-arrival and service times constantly 1.4 and 3 minutes respectively, explain discrete simulation technique taking 10 minutes as the simulation period. Find from this the average waiting time of a customer (assume that initially the system is empty and the first customer arrives at $t = 0$).
2. At the telephone booth, suppose that the customers arrive with an average time of 1.2 time units between an arrival and the next. Service times are assumed to be 2.8 time units. Simulate the system for 12 time units by assuming that the system starts at $t = 0$. What is the average waiting time per customer ?
3. suppose there are two types of customers. Type *A* customer have priority for service so that those of type *B* cannot be serviced until all waiting (type) customer have completed their service. The service time for type *A* and type *B* customer are 3 and 1, respectively. Further assume that the inter-arrival times for type *A* and *B* are 1.5 and 3 respectively. Simulate the system for 15 time units. Compute thd average number of waiting customer for each type as well as the combined average. What is the average waiting time per each type customer ?

   (Assume that the system starts at $t = 0$).

17.9. GENERATION OF RANDOM NUMBERS

Random Variable. *The random variable is a real valued function defined over a sample space associated with the outcome of a conceptual chance experiment.*

Random variables are classified according to their probability density function.

Random Variate. *It refers to a particular outcome of an experiment, i.e., a numerical or sample value of a random variable.*

Random Number. *As generally understood it refers to a uniform random variable or a numerical value assigned to a random variable following uniform probability density function (i.e., normal, Poisson, exponential, etc.). In other words, it is a number in a sequence of numbers whose probability of occurrence is the same as that of any other number in that sequence.*

Pseudo-random Numbers. *Random numbers are called pseudo-random numbers when they are generated by some deterministic process but they qualify the pre-determined statistical test for randomness.*

The sequence of numbers generated by such process is completely determined by the input data (or the first random number) used for the method.

Generation of Random Numbers. *Monte-Carlo simulation* needs the generation of a sequence of random numbers which constitute an integral part of the simulation model and also help in determining random observations from the probability distribution. Random numbers are assigned in such a manner that their proportion is exactly equal to the probability distribution. Random numbers may be found by computer, by random tables, or manually.

Generally, it is not practical to obtain random numbers manually because it requires much labour. Usually such devices as : *roulette wheels, dice rolling, card shuffling, etc.*, are used in these methods.

But the most common method to obtain random numbers is to generate them by a computer program. These numbers lie between 0 and 1 (0 and 100%) in conjunction with the cumulative probability distribution of a random variable, including 0, but not 1.

Many tables of random numbers are available in the literature. These numbers are considered to be truly random numbers because these were generated using some random physical process.

While choosing the random numbers from the table, the starting point on the table is immaterial. We may start with any number in any column or row, and proceed in the same column or row to the next number, but a consistent, unvaried pattern should be used in drawing random numbers. We should not jump from one number to another indiscriminately.

In the case of choosing random numbers for more than one concerned variable, different list of numbers for each variable should be used, otherwise same random numbers would imply dependence among different variables.

Mid-Square Method of Generating Pseudo-Random Numbers :

This is one of the commonly used methods to generate sequences of pseudo-random numbers. The method works as follows :

Select a four-digit integer or seed to initialise the generator. The first random number is obtained from the seed in the following manner :

The seed is squared, the resulting number is supposed to contain 8 digits (if less digits are there, leading zeros are inserted). From this number, the middle four digits are extracted as the required random number. This number is subsequently used as the new seed. Pseudo-random numbers are generated in this manner, each time using the previous random number as the new seed.

For example, consider the first seed as 2714. By squaring 2714, we get the number 07365796.

From this number four middle digits 3657 are extracted which is the first random number. To get the next random number, we square 3657 to get 13373649. The next random number is 3736. This process is repeated until the required number of random numbers are obtained.

Limitations of Mid-Square Method :

1. The method tends to degenerate rapidly. A random number may reproduce itself.
   For example, $x_9 = 7600$, $x_{10} = 7600$, $x_{11} = 7600$.
2. If the number zero is ever generated, all subsequent numbers generated will also have a zero value unless steps are provided to handle this case.
3. A loop may generate, *i.e.* the same sequence of random numbers can repeat.
   For example, $x_{15} = 6100$, $x_{16} = 2100$, $x_{17} = 4100$, $x_{18} = 8100$, $x_{19} = 6100$, $x_{20} = 2100$, $x_{21} = 4100$.
5. This method is slow since many multiplications and divisions are required to access the middle digits in a fixed word binary computer.

Q. What are Pseudo-random numbers ? Describe the mid-square method of generating Pseudo-random numbers. What are its limitations ?

17.10. MONTE-CARLO SIMULATION

The *Monte-Carlo technique* has become so much important part of simulation models that the terms are often assumed to be synonymous. However, it is only a special technique of simulation. The technique of *Monte-Carlo* involves the selection of random observations within the simulation model.

This technique is restricted for application involving random numbers to solve *deterministic* and *stochastic* problems. The principle of this techinque is replacement of actual statistical universe by another universe described by some assumed probability distribution and then sampling from this theoretical population by means of random numbers.

In fact, this process is the generation of simulated statistics (random variables) that can be explained in sample terms as choosing a random number and substituting this value in standard probability density function to obain random variable or simulated statistics.

Example 5. ***(The Chef Example).*** *The number of customers at a restaurant each evening is distributed as shown below :*

Numer of customers	*Lots*	*Average*	*Very few*
Probability	0.2	0.4	0.4